

A differential geometric setting for BRS transformations and anomalies I

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Abstract. *The Becchi-Rouet-Stora relations and the cohomological construction of anomalies of gauge fields are described within the theory of principal connections of smooth vector bundles.*

INTRODUCTION

This paper provides a detailed introduction to the differential geometric and cohomological framework underlying BRS transformations and anomalies of gauge fields. These items both appeared in the study of perturbative renormalization of gauge fields – renormalization being required to yield in particular physical answers independent of the choice of gauge [3]. The vanishing of anomalies thus appears as a criterion for the relevance of fundamental field theories (1). However anomalies play also a positive role in a different context, that of phenomenological theories for the search of which they provide a means of writing «effective lagrangians». It is in this context that anomalies were first found [11]. Though they arose in a quantum (field theory) context, BRS transformations and anomalies ultimately appear as purely classical (differential geometric)

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(1) See, however, [10] where it is suggested that gauge theories with anomalies may have a consistent interpretation at the non-perturbative level.

objects, which can be isolated as such from the original quantum context – this is what we do in the present paper. In fact the anomalies themselves – and the algorithms which are useful for their description – are elements of certain vector valued Lie algebra cohomologies related to the ambient Yang-Mills principal bundle. This fact was realized following the discovery of the Wess-Zumino compatibility condition, which in fact characterizes 1-cocycles of the cohomology of the Lie algebra of the gauge group with values «local» functionals of the potentials (connection one-forms).

Our paper comprises seven sections, amongst which sections 1, 2, 3 and 5 describe prerequisites to the actual subject matter in sections 4, 6 and 7. We included these prerequisites in order to be complete, and also because of the necessity of fixing notation. Section 1 describes the De Rham complex Λ^* of a principal bundle P with values in the symmetric tensors on the Lie algebra L of the structure group G , and defines the commuting actions, on this De Rham complex, of G and of the gauge group \mathcal{G} . Section 2 describes the subcomplex A^* of fixpoints of the action of G (i.e. Ad-equivariant element of Λ^*). Section 3 describes the cohomology algebra of the Lie algebra \mathcal{L} of the gauge group. After these prerequisites, Section 4 describes the cohomology of \mathcal{L} with values in A^* , with the ensuing double complex and differential algebra structures. This furnishes the framework of the BRS relations, to which section 5 is devoted, as well as a framework for the construction of anomalies, described in section 7. The cohomology algebra of \mathcal{L} with values in local functionals of connection one-forms – the receptacle for anomalies – is defined in section 6.

This expository paper leaves aside important aspects to which we shall return later, e.g.

(i) The additional analytical apparatus arising from the fact, realized in physics, that the structure group G is a linear group (a group of matrices – its Lie algebra L consisting then also of matrices). We here look at G as a general Lie group, in the spirit of the general theory of smooth principal bundle.

(ii) The Chevalley cohomology of the gauge group Lie algebra \mathcal{L} with values in the local functionals of the potentials (resp., in a $S(L)$ -valued De Rham complex) has a version utilizing equivariant differential forms on the gauge group \mathcal{G} himself, with the operators stemming from the exterior derivative of \mathcal{G} .

(iii) The homotopy formula should be generalized in two respects: one can avoid the assumption of triviality of the principal bundle at hand by introducing a background field. On the other hand, it is useful to consider multidimensional generalizations of «transgression» involving more than one potential. This necessitates the replacement of $S(L)$ by an algebra of «graded symmetric» forms.

§1. THE GAUGE GROUP \mathcal{G} OF A SMOOTH PRINCIPAL BUNDLE

$\mathbf{P} = (\mathbf{P}^* \rightarrow \mathbf{M}, G)$. ACTIONS OF G AND \mathcal{G} ON THE REAL AND VECTOR-

-VALUED DE RHAM ALGEBRAS $(\Lambda^*(P, \mathbb{R}), d, \cdot)$, $(\Lambda^*(P, S(L)), d, \times)$, AND $(\Lambda^*(P, L), d, [\cdot])$

Our basic object in this paper is a smooth principal bundle $\mathbf{P} : P^* \rightarrow M$, with basis M and (compact) structural Lie group G . We shall denote L the Lie algebra of G , and write $[u, v]$ for the Lie bracket of $u, v \in L$. We denote by R the right action of G on P :

$$(1.1) \quad R_s z = zs, \quad s \in G, \quad z \in P.$$

[1.1] The gauge group \mathcal{G} and its Lie algebra \mathcal{L}

The gauge group \mathcal{G} is the group of automorphisms of \mathbf{P} inducing the identity on M . Specifically \mathcal{G} consists of the diffeomorphisms: $P \rightarrow P$ commuting with all $R_s, s \in G$, and mapping each fiber into itself. Since Ψ acts on the fiber, we have

$$(1.2) \quad \Psi(z) = zg(z), \quad z \in P,$$

where g is a smooth map: $P \rightarrow G$, *ad-equivariant* in the sense

$$(1.3) \quad g(zs) = \text{Ad } s^{-1}(g(z)) = s^{-1}g(z)s, \quad z \in P, \quad s \in G,$$

this expressing commutativity of Ψ and R_s . Relation (1.2) in fact establishes a bijection between the elements Ψ of \mathcal{G} and the smooth *ad-equivariant* maps $g : P \rightarrow G$, whereby products and inverses in \mathcal{G} are turned into pointwise products, resp. inverse:

$$(1.4) \quad \begin{cases} \Psi \leftrightarrow g \\ \Psi' \leftrightarrow g' \end{cases} \Rightarrow \begin{cases} \Psi^{-1} \leftrightarrow g^{-1}, & g^{-1}(z) = g(z)^{-1} \\ \Psi\Psi' \leftrightarrow gg', & (gg')(z) = g(z)g'(z) \end{cases}, \quad z \in P.$$

\mathcal{G} is an «infinite dimensional Lie group» (a *diffeological group* in the sense of Souriau [13]). As such it possesses a Lie algebra which we denote \mathcal{L} . We can view \mathcal{L} as the set of smooth maps $\Omega : P \rightarrow L$. *Ad-equivariant* in the sense:

$$(1.5) \quad \Omega(zs) = \text{Ad } s^{-1}(\Omega(z)) = \ll s^{-1}\Omega(z)s, \gg \quad s \in G$$

(here $\text{Ad } s$ is the tangent map of $\text{ad } s = s \cdot s^{-1}$ at the unit of G). Lie bracket and exponential are then obtained pointwise:

$$(1.6) \quad [\Omega, \Omega'](z) = [\Omega(z), \Omega'(z)], \quad \Omega, \Omega' \in \mathcal{L}, \quad z \in P$$

(left hand side bracket in \mathcal{L} , right hand side bracket in L),

$$(1.7) \quad e^{\Omega}(z) = e^{\Omega(z)}, \quad \mathcal{O} \in \mathcal{L}, \quad z \in P.$$

Setting, for $\psi \in \mathcal{G}^\infty(M)$, $(\psi\Omega)(z) = \psi(\pi(z))\Omega(z)$, $\Omega \in \mathcal{L}, z \in P$, we thus obtain

an action of $\mathcal{C}^\infty(M)$ on \mathcal{L} commuting with Lie-brackets; thus \mathcal{L} is a Lie algebra over $\mathcal{C}^\infty(M)$.

[1.2]. The (real valued) De Rham complex $(\Lambda^*(P, \mathbb{R}), d, \wedge)$ as a GCDA. Action of the Lie algebra $\mathcal{X}(P)$ on $\Lambda^*(P, \mathbb{R})$. Representation of G and \mathcal{G} on $\Lambda^*(P, \mathbb{R})$

We write $\Lambda^*(P, \mathbb{R}) = \oplus \Lambda^p(P, \mathbb{R})$, with $\Lambda^*(P, \mathbb{R})$ the set of smooth real-valued differential p -forms on P . Denoting by $\mathcal{X}(P)$ the Lie algebra of smooth vector fields on P , we can view $\Lambda^*(P, \mathbb{R})$ as the set of $\mathcal{C}^\infty(P)$ -valued, alternate $\mathcal{C}^\infty(P)$ -linear p -forms on $\mathcal{X}(P)$ ($\mathcal{C}^\infty(P) = \Lambda^0(P, \mathbb{R})$). The wedge product \wedge , exterior derivative d Lie derivative $L(\xi)$ along $\xi \in \mathcal{X}(P)$ and inner product $i(\xi)$ by $\xi \in \mathcal{X}(P)$, are then defined as follows: for $\xi_0, \xi_1, \dots, \xi_{p+q} \in \mathcal{X}(P)$, $\alpha \in \Lambda^*(P, \mathbb{R})$, $\beta \in \Lambda^*(P, \mathbb{R})$ we have

$$(1.8) \quad (\alpha \wedge \beta)(\xi_1, \dots, \xi_{p+q}) = \\ = \frac{1}{p!} \frac{1}{q!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma) \alpha(\xi_{\sigma_1}, \dots, \xi_{\sigma_p}) \beta(\xi_{\sigma_{p+1}}, \dots, \xi_{\sigma_{p+q}})$$

$$(1.9) \quad (d\alpha)(\xi_0, \dots, \xi_p) = \sum_{i=0}^p (-1)^i \xi_i \{ \alpha(\xi_1, \dots, \hat{\xi}_i, \dots, \xi_p) \} + \\ + \sum_{0 < i < j < p} (-1)^{i+j} \alpha([\xi_i, \xi_j], \xi_0, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_p)$$

$$(1.10) \quad \{L(\xi)\alpha\}(\xi_1, \dots, \xi_p) = \xi(\alpha(\xi_1, \dots, \xi_p)) - \\ - \sum_{i=1}^p \alpha(\xi_1, \dots, \xi_{i-1}, [\xi, \xi_i], \xi_{i+1}, \dots, \xi_p)$$

$$(1.11) \quad \left\{ \begin{array}{l} \langle i(\xi)\alpha \rangle(\xi_1, \dots, \xi_{p-1}) = \alpha(\xi, \xi_1, \dots, \xi_{p-1}) \\ i(\xi) = 0 \quad \text{on} \quad \Lambda^0(P, \mathbb{R}). \end{array} \right.$$

Through definitions (1.8), (1.9) $(\Lambda^*(P, \mathbb{R}), d, \wedge)$ now becomes a GCDA. And definitions (1.10) and (1.11) determine an action (L, i) of the Lie algebra $\mathcal{X}(P)$ on this GCDA. For a proof of these well known facts we refer e.g. to [1], Corollary [10].

In addition to the previous structure the De Rham complex $\Lambda^*(P, \mathbb{R})$ is both a G -space and a \mathcal{G} -space. We obtain the action of $s \in G$ on $\Lambda^*(P, \mathbb{R})$ as $r(s) = = R_s^* \alpha$, where $R_s^* \alpha$ denotes the pull back of the differential form α by R_s ,

specifically (2):

$$(1.12) \quad \{r(s)\alpha\}(z, Z_i) = \alpha(zs, R_{s*} Z_i), \quad z \in P, \quad Z_i \in T_z^P.$$

In this way G is represented in the zero grade automorphisms of the GCDA $(\Lambda^*(P, \mathbb{R}), d, \wedge)$ (indeed $r(s)$, $s \in G$, commutes with d and with the wedge product). The corresponding representation of L :

$$(1.13) \quad \theta(u) = \left. \frac{d}{dt} \right|_{t=0} R_{e^{tu}}^*, \quad u \in L,$$

then arises as the composition

$$(1.14) \quad \theta(u) = L(\xi^u), \quad u \in L,$$

with ξ^u the principal field

$$(1.15) \quad \xi_z^u = L_{z*} e^u, \quad u \in L,$$

(here L_z , $z \in P$, is the map: $G \rightarrow P$ determined by

$$(1.16) \quad L_z s = zs (= R_s z), \quad z \in P, \quad s \in G.$$

From this and the convention

$$(1.17) \quad i(u) = i(\xi^u), \quad u \in L,$$

we get an action (θ, i) of the Lie algebra L on $\Lambda^*(P, \mathbb{R})$.

We now describe the representation ρ of \mathcal{G} on $\Lambda^*(P, \mathbb{R})$: for $\Psi \in \mathcal{G}$, $\rho(\Psi)$ is obtained by pulling back the (inverse action of \mathcal{G})⁽³⁾

$$(1.18) \quad \rho(\Psi)\alpha = \Psi^{-1*}\alpha, \quad \Psi \in \mathcal{G}, \quad \alpha \in \Lambda^*(P, \mathbb{R}),$$

specifically we have (4)

$$(1.18a) \quad \{\rho(\Psi)\alpha\}(z, Z_i) = \alpha(\Psi^{-1}(z), (\Psi^{-1})_{*z} Z_i), \quad z \in P, \quad Z_i \in T_z^P.$$

We shall also denote ρ the accompanying representation of \mathcal{L} :

$$(1.19) \quad \rho(\Omega) = \left. \frac{d}{dt} \right|_{t=0} \rho(e^{t\Omega}), \quad \Omega \in \mathcal{L}.$$

(2) Since both the right action and the pull back are product-inverting, we obtain indeed a representation $s \rightarrow R_s^*$ of G on $\Lambda^*(P, \mathbb{R})$. For a definition in terms of sections are [1.5] below.

(3) Since \mathcal{G} acts on P on the left, we now have to pull back the inverse of Ψ .

(4) For a definition in terms of sections, see [1.5] below.

Specifically, one has, for $\Psi = e^{t\Omega}$

$$(1.20) \quad (\Psi^{-1})_{*z} = (R_{e^{-t\Omega(z)}})_{*z} - t(L_{ze^{-t\Omega(z)}})_{*e} d\Omega(z, \cdot)$$

\mathcal{G} (resp. \mathcal{L}) are thereby represented in the zero grade automorphisms (resp. derivatives commuting with d) of the GCDA $(\Lambda^*(P, \mathbb{R}), d, \cdot)$ (immediate consequence of the commutativity of the pull back (1.18) with d and with the wedge product). Furthermore, since $\Psi \in \mathcal{G}$ commutes with $R_s, s \in G$, the representations of G and \mathcal{G} on $\Lambda^*(P, \mathbb{R})$ commute.

So much about real-valued differential forms on P . We now describe the differential forms on P with values in L , or more generally in symmetric tensors over L .

[1.3]. The $S(L)$ -valued De Rham complex $(\Lambda^*(P, S(L)), d, \alpha)$ as GCDA. Representations of G and \mathcal{G} on $\Lambda^*(P, S(L))$

We denote by $S(L)$ the symmetric algebra over L :

$$(1.21) \quad \begin{cases} S(L) = \bigoplus_{k \in \mathbb{N}} S_k(L) \\ S_k(L) = L^{\vee k} = S_k L^{\otimes k}, S_0(L) = \mathbb{R} \end{cases}$$

equipped with the symmetric product

$$(1.21a) \quad f \smile f' = S_{k+i}(\varphi \otimes \varphi'), \quad f \in S_k(L), f' \in S_i(L).$$

Here $S_k, k > 0$ the idempotent projecting $L^{\otimes k}$ onto the symmetric tensors – vanishing on the $f \otimes f' - (-1)^{i+j} f' \otimes f, i, j \in \mathbb{N}, i+j=k$. And $S_0 = \text{id}_L$. Note that the dual $S_k^*(L)$ can be identified with the set of symmetric k -linear forms P on L ⁽⁵⁾ by writing

$$(1.22) \quad P(u_1, \dots, u_k) = P(u_1 \smile \dots \smile u_k), \quad u_s, \dots, u_k \in L.$$

A subset of $S_k^*(L)$ of particular interest is the subset $I_k(L)$ of *Ad-invariant symmetric k -forms* characterized by

$$(1.23) \quad P(\text{Ad } s(u_1), \dots, \text{Ad } s(u_k)) = P(u_1, \dots, u_k), \quad s \in G, u_1, \dots, u_k \in L$$

or equivalently

$$(1.23a) \quad \sum_{i=1}^n P(u_1, \dots, u_{i-1}, [u, u_i], u_{i+1}, \dots, u_k) = 0, \quad u, u_1, \dots, u_k \in L.$$

⁽⁵⁾ The latter are in turn one-to-one with the *polynomials of order k on L* , the passage from k -forms to polynomials arising by restriction to the diagonal, and inversely by polarization.

Now we consider the $S(L)$ -valued De Rham complex of P :

$$(1.24) \quad \Lambda^*(P, S(L)) = \bigoplus_P \Lambda^P(P, S(L))$$

with $\Lambda^P(P, S(L))$ the set of $S(L)$ -valued smooth differential forms on P , alternatively

$$(1.25) \quad \Lambda^P(P, S(L)) = S(L) \otimes \Lambda^P(P, \mathbb{R})$$

with the identification

$$(1.25a) \quad \left\{ \begin{array}{l} \tau = f \otimes \alpha \Leftrightarrow \langle \tau(\xi_1, \dots, \xi_p) = \alpha(\xi_1, \dots, \xi_p) f, \text{ for all } \xi_1, \dots, \xi_p \\ \tau \in \Lambda^P(P, S(L)), \quad \alpha \in \Delta^P(P, \mathbb{R}), \quad f \in S(L). \end{array} \right.$$

Of course $\Lambda^P(P, S(L))$ decomposes into «homogeneous components» ⁽⁶⁾:

$$(1.26) \quad \left\{ \begin{array}{l} \Lambda^P(P, S(L)) = \bigoplus_{k \in \mathbb{N}} \Lambda_k^P(P, S(L)) \\ \Lambda_k^P(P, S(L)) = \Lambda^P(P, S_k(L)) = \Lambda^P(P, \mathbb{R}) \otimes S_k(L) \end{array} \right.$$

the term $k = 0$ arising from our convention $S_0(L) = \mathbb{R}$, which implies

$$(1.27) \quad \Lambda_0^P(P, S(L)) = \Lambda^P(P, \mathbb{R})$$

(incorporating in this way the real valued De Rham complex has a convenient unifying virtue).

On $\Lambda^*(P, S(L))$ we now define (7)

– a bilinear product \times by requiring

$$(1.28) \quad (f \otimes \alpha) \times (f' \otimes \alpha') = (f \sim f') \otimes (\alpha \sim \alpha'), \left\{ \begin{array}{l} \alpha, \alpha' \in \Lambda(P, \mathbb{R}) \\ f, f' \in S(L) \end{array} \right.$$

– operators, $d, i(u) \theta(u), u \in L$ as follows:

$$(1.29) \quad d = \text{id}_{S(L)} \otimes d$$

⁽⁶⁾ We accordingly write $\Lambda_k^*(P, S(L)) = \bigoplus_P \Lambda_k^P(P, S(L))$.

⁽⁷⁾ Note that these definitions extend $d, i(u)$ and $\rho(\Psi)$ by requiring them to be trivial on $S(L) = S(L) \otimes 1, 1$ the unit function on P . In contrast $\theta(u)$ and $r(s)$ are obtained from tensorizing with the adjoint representation. This will result in a departure from a «Lie action situation», see (ii) below.

$$(1.30) \quad i(u) = \text{id}_{S(L)} \otimes i(u)$$

$$(1.31) \quad \theta(u) = \text{id}_{S(L)} \otimes \theta(u) + \text{Ad } u \otimes \text{id}_{\wedge^p(P, \mathbb{R})}$$

(with the following definition of $\text{Ad } u$ on $S(L)$)

$$(1.32) \quad \begin{cases} \text{Ad } u(u_1 \sim \dots \sim u_k) = \sum_{i=1}^k u_1 \sim \dots \sim u_{i-1} \sim [u, u_i] \sim u_{i+1} \sim \dots \sim u_k \\ u_1, \dots, u_k \in L \end{cases}$$

– actions r and ρ of G , resp. \mathcal{F} , as follows:

$$(1.33) \quad r(s) = \text{Ad } s \otimes R_s^*, \quad s \in G$$

(with the following definition of $\text{Ad } s$ on $S(L)$):

$$(1.34) \quad \text{Ad } s(u_1 \sim \dots \sim u_k) = \begin{cases} \text{Ad } s(u_1) \sim \dots \sim \text{Ad } s(u_k), & u_1, \dots, u_k \in L \\ \text{Ad } s = \text{id} & \text{on } S_0(L) \end{cases}$$

$$(1.35) \quad \rho(\Psi) = \text{id}_{S(L)} \otimes \rho(\Psi)$$

the latter yielding as usual

$$(1.36) \quad \rho(\Omega) = \frac{d}{dt} \Big|_{t=0} \rho(e^{t\Omega}), \quad \Omega \in \mathcal{L}.$$

We note that these definitions reduce on $\Lambda_0^*(P, S(L))$ to our former definitions (1.8), (1.9), (1.13), (1.17), (1.12) and (1.18), in accordance with embedding (1.27).

The foregoing definitions now imply the following:

(i) $\Lambda^*(P, S(L), d, \times)$ is a GCDA whose sub-GCDA $(\Lambda_0^*(P, S(L), d, \times)$ is isomorphic with $(\Lambda^*(P, \mathbb{R}), d, \cdot)$.

(ii) The pair (θ, i) behaves as an action of $\mathcal{X}(P)$ on this GCDA, but for the fact that one has ⁽⁸⁾

$$(1.37) \quad i(u)d + di(u) = \text{id}_{S(L)} \otimes \theta(u), \quad (= \theta(u) - \text{Ad } u \otimes \text{id}_{\wedge^p(P, \mathbb{R})}), \quad u \in L.$$

⁽⁸⁾ In other terms, with the replacement $\alpha \rightarrow \Lambda^*(P, S(L))$, the product $\cdot \rightarrow$ the product \times , one has properties (A.1) through (A.16) in Appendix A except property (A.14) to be replaced by (1.37).

(iii) r as defined in (1.33) is a representation of G on $\Lambda^*(P, S(L))$ by zero-grade automorphisms of the latter as a GCDA moreover such that the accompanying representation of L coincides with θ (9):

$$(1.38) \quad \frac{d}{dt} \Big|_{t=0} r(e^{tu}) = \theta(u), \quad u \in L$$

(iv) ρ as defined in (1.35) (resp. (1.36)) is a representation of \mathcal{G} (resp. \mathcal{L}) on $\Lambda^*(P, S(L))$ by zero grade automorphisms of the latter as a GCDA (resp. by derivations of $(\Lambda^*(P, S(L)), \cdot)$ commuting with d)

(v) The representations r and ρ commute (10):

$$(1.39) \quad \rho(\Psi)r(s) = r(s)\rho(\Psi), \quad s \in G, \quad \Psi \in \mathcal{G},$$

accordingly

$$(1.40) \quad \rho(\Omega)\theta(u) = \theta(u)\rho(\Omega), \quad u \in L, \quad \Omega \in \mathcal{L}.$$

These facts are classical. For a proof of (i) we refer to, e.g. [1, Theorem 1.8] with the replacement $L \rightarrow \mathcal{X}(P)$, $A \rightarrow \mathcal{C}^\infty(P)$, $V \rightarrow S(L) \otimes \mathcal{C}^\infty(P)$, $\rho(\xi) \rightarrow \text{id}_{S(L)} \otimes \xi$, $d_\rho \rightarrow d$, $\cdot \rightarrow \times$. The proof of (ii) is as follows: denote $\theta_1(u)$, resp. $\theta_2(u)$ the first, resp. second term r.h.s. of (1.31): (θ_1, i) is a bona fide action of the Lie algebra L on the GCDA $\Lambda^*(P, S(P))$, obtained by tensoring by $\text{id}_{S(L)}$ the action (θ, i) of L on $\Lambda^*(P, \mathbb{R})$. We examine the changes in (A.9), (A.12) through (A.16) brought about by the change $\theta_1 \rightarrow \theta_1 + \theta_2$. Since $\theta_2(u)$ is a zero grade derivation, (A.9) stays unchanged. Since θ_1 , and θ_2 are mutually commuting representation of the Lie algebra L , (A.12) is maintained.

We check (A.13), from which (A.16) follows: we have, from (1.30), (1.31)

$$(1.41) \quad \begin{aligned} \theta(u)i(v) - i(v)\theta(u) &= \text{id}_{S(L)} \otimes \{(\theta_1(u)i(v) - i(v)\theta(u))\} = \\ &= \text{id}_{S(L)} \otimes i([u, v]) = i([u, v]). \end{aligned}$$

We examine (A.14): we have, from (1.21), (1.30):

$$(1.42) \quad \begin{aligned} i(u)d + di(u) &= \text{id}_{S(L)} \otimes (i(u)d + di(u)) = \\ &= \text{id}_{S(L)} \otimes \theta(u). \end{aligned}$$

Finally we have from (1.29), (1.31)

(9) Coherent with the fact that the $\theta(u)$ are derivations commuting with d (cf. A.13) and (A.15).

(10) We could thus consider $G \times \mathcal{G} \ni (s, \Psi) \rightarrow r(s)\rho(\Psi)$ as a representation of the direct product of the groups G and \mathcal{G} .

$$(1.43) \quad d\theta(u) - \theta(u)d = \text{id}_{S(L)} \otimes (d\theta(u) - \theta(u)d) = 0$$

hence (A.15) stays unchanged.

Remark. Defining $L(\xi)$ and $i(\xi)$, $\xi \in \mathcal{X}(P)$ on $\Lambda^*(P, S(L))$ as

$$(1.44) \quad L(\xi) = \text{id}_{S(L)} \otimes L(\xi)$$

$$(1.45) \quad i(\xi) = \text{id}_{S(L)} \otimes i(\xi)$$

we get an action of the Lie algebra $\mathcal{X}(M)$ on the GCDA $\Lambda^*(P, S(L))$ yielding the above action (θ_1, i) by composition with ξ in (1.15).

[1.4]. Our next concern is the «descent» from $S_k(L)$ -valued to real differential forms by means of elements of $S_k^*(L)$. Consider a «polynomial» $P \in S_k^*(L)$ of order k : taking its covalue with a p -form $\tau \in \Lambda_k^p(P, S(L))$:

$$(1.46) \quad P(\tau)(\xi_1, \dots, \xi_p)(z) = P(\tau(\xi_1, \dots, \xi_p)(z)), \quad \xi_1, \dots, \xi_p \in \mathcal{X}(P), \quad z \in P$$

(in other terms:

$$(1.46a) \quad P(f \otimes \alpha) = P(f)\alpha, \quad \alpha \in \Lambda^p(P, \mathbb{R}), \quad f \in S_k(L))$$

one gets an element of $\Lambda^p(P, \mathbb{R})$. We can thus view the dual $S_k^*(L)$ of $S_k(L)$ as providing linear maps

$$(1.47) \quad P : \Lambda_k^p(P, S(L)) \ni \tau \rightarrow P(\tau) \in \Lambda_k^p(P, \mathbb{R}),$$

with the properties

$$(1.48) \quad P \circ d = d \circ P$$

$$(1.49) \quad P \circ i(u) = i(u) \circ P, \quad u \in L$$

$$(1.50) \quad P \circ r(s) = r(s) \circ P \circ \text{Ad } s, \quad s \in G.$$

Proof. (1.48), (1.49) follow from (1.46), (1.29), (1.30):

$$(1.51) \quad \begin{aligned} P \circ d(f \otimes \alpha) &= P(f \otimes d\alpha) = P(f)d\alpha = \\ &= d(P(f)\alpha) = d \circ P(f \circ \alpha) \end{aligned}$$

$$(1.52) \quad \begin{aligned} P \circ i(u)(f \otimes \alpha) &= P(f \otimes i(u)\alpha) = P(f)i(u)\alpha = \\ &= i(u)(P(f)\alpha) = i(u) \circ P(f \circ \alpha) \end{aligned}$$

On the other hand, (1.50) follows from (1.33), (1.46) (cf. (1.23))

$$\begin{aligned}
 (1.53) \quad P \circ r(s)(f \otimes \alpha) &= P(\text{Ad } s(f) \otimes R_s^* \alpha) = \\
 &= P(\text{Ad } s(f)) \circ R_s^* \alpha = \\
 &= R_s^* P(\text{Ad } s(f)) \cdot \alpha = \\
 &= r(s)(P \circ \text{Ad } s)(f \otimes d).
 \end{aligned}$$

The 1-tensor part $\Lambda^*(P, L)$ of $\Lambda^*(P, S(L))$ deserves a special examination, since it inherits from the Lie bracket of L a graded Lie algebra structure (essential for expressing the BRS relations).

[1.5]. The L-valued De Rham complex $(\Lambda^*(P, L), d, [\cdot])$ as a DGL

Recalling the identification

$$(1.54) \quad \left\{ \begin{aligned}
 \Lambda_1^p(P, S(L)) &= \Lambda^p(P, L) = L \otimes \Lambda^p(P, \mathbb{R}) \\
 \lambda &= u \otimes \alpha \Leftrightarrow \lambda(\xi_1, \dots, \xi_p) = \alpha(\xi_1, \dots, \xi_p)u \\
 \lambda &\in \Lambda^p(P, L), \alpha \in \Lambda^p(P, \mathbb{R}), \xi_1, \dots, \xi_p \in \mathcal{X}(P)
 \end{aligned} \right.$$

we define as follows the Schouten product $[\cdot]$ on $\Lambda^*(P, L)$: for $\lambda \in \Lambda^p(P, L)$, $\mu \in \Lambda^q(P, L)$, $p, q \in \mathbb{N}$, $\xi_1, \dots, \xi_{p+q} \in L$ we set

$$\begin{aligned}
 (1.55) \quad [\lambda \wedge \mu](\xi_1, \dots, \xi_{p+q}) &= \\
 &= \frac{1}{p!} \frac{1}{q!} \sum_{\sigma \in \Sigma_{p+q}} \chi(\sigma) [\lambda(\xi_{\sigma_1}, \dots, \xi_{\sigma_p}), \mu(\xi_{\sigma_{p+1}}, \dots, \xi_{\sigma_{p+q}})]
 \end{aligned}$$

where $[\cdot, \cdot]$ s.h.s. denotes a Lie bracket in L . Alternative specification:

$$(1.55a) \quad [u \otimes \alpha \wedge v \otimes \beta] = [u, v] \otimes (\alpha \wedge \beta), \quad \left\{ \begin{aligned}
 \alpha, \beta &\in \Lambda^*(P, \mathbb{R}) \\
 u, v &\in L
 \end{aligned} \right.$$

With this definition, we have that $(\Lambda^*(P, L), d, [\cdot])$ is a DGL. Moreover r , resp. ρ , restricted to $\Lambda^*(P, L)$ are commuting representations of the group G , resp. \mathcal{G} , by zero-grade automorphisms of $(\Lambda^*(P, L), d, [\cdot])$. Correlatively, θ , resp., ρ are commuting representations of the Lie algebras L , resp. \mathcal{L} by zero grade derivations of $(\Lambda^*, [\cdot])$ commuting with d . These facts are classical. A proof of the DGL property of $(\Lambda^*(P, L), d, [\cdot])$ can be inferred from [1, Theorem 1.8] with the replacement $L \rightarrow \mathcal{X}(P)$, $A \rightarrow \mathcal{C}^\infty(P)$, $V \rightarrow \mathcal{C}^\infty(P) \otimes L$, $\rho(\xi) = \xi \otimes \text{id}_L$, $d_p \rightarrow d$, $\cdot \rightarrow [\cdot]$. We know from the preceding paragraph that $r(s)$ and $\rho(\Psi)$, $s \in G$, $\Psi \in \mathcal{G}$, commute with d . Moreover (1.48a) shows that they commu-

te with the Schouten product: indeed we saw that R_s^* and $\rho(\Psi)$ acting on $\Lambda^*(P, \mathbb{R})$ commute with the wedge product. And $\text{Ad } s$ commutes with the Lie bracket of L :

$$(1.56) \quad [\text{Ad } s(u), \text{Ad } s(v)] = \text{Ad } s([u, v]), \quad s \in G, \quad u, v \in L.$$

[1.6]. The groups G and \mathcal{G} as acting on sections

We mentioned that $\Lambda^*(P, \mathbb{R})$ can be considered as the set of $\mathcal{C}^\infty(P)$ -valued, alternate, $\mathcal{C}^\infty(P)$ -linear p -forms on $\mathcal{X}(P)$ and formulated the definitions (1.8) through (1.11) and (1.14), (1.17) in this context. We here give for completeness the corresponding definitions of the representations r and ρ . Letting G and \mathcal{G} act on $\mathcal{X}(P)$ ⁽¹¹⁾ as follows:

$$(1.57) \quad \{r(s)\xi\}_z = (R_s)_* \xi_{zs^{-1}}, \quad s \in G$$

$$(1.58) \quad \{\rho(\Psi)\xi\}_z = (\Psi^{-1})_* \xi_{\Psi(z)}, \quad \Psi \in \mathcal{G}$$

the definitions (1.33), (1.35) are alternatively phrased as follows: one has, for $\tau \in \Lambda^p(P, S(L))$

$$(1.33a) \quad r(s)\tau(\xi_1, \dots, \xi_p) = \text{Ad } s(\tau(r(s)\xi_1, \dots, r(s)\xi_p) \circ R_s), \quad s \in G$$

$$(1.35a) \quad (\rho(\Psi)\tau)(\xi_1, \dots, \xi_p) = \tau(\rho(\Psi)\xi_1, \dots, \rho(\Psi)\xi_p) \circ \Psi^{-1}. \quad \Psi \in \mathcal{G}$$

It then follows from

$$(1.59) \quad r(s)\xi_u = \xi_{\text{Ad } s^{-1}(u)}, \quad u \in L, \quad s \in G$$

that one has

$$(1.60) \quad r(s)i(u) = i(\text{Ad } s(u))r(s), \quad u \in L, \quad s \in G.$$

§2. THE DIFFERENTIAL SUBALGEBRAS. (A^*, d, \cdot) , (A_0^*, d, \mathbf{x}) AND $(A_1^*, d, [\cdot])$ OF REAL INVARIANT, RESP. $S(L)$ AND L -VALUED AD-EQUIVARIANT DIFFERENTIAL FORMS

The fixed point set, for the action of G , of the differential algebras of the last sections, are differential subalgebras of direct relevance for gauge-field theory. We devote this section to their description.

[2.1]. The GCDA (A^*, d, \mathbf{x}) of Ad-equivariant elements of $\Lambda^*(P, S(L))$

We say that $\tau \in \Lambda^*(P, S(L))$ is *Ad-equivariant* whenever one has

(11) On the right.

$$(2.1) \quad \left\{ \begin{array}{l} R_s^* \tau = \text{Ad } s^{-1} \tau \quad \text{for all } s \in G \\ \text{i.e. } \tau(zs, R_{g^* z} Z_i) = \text{Ad } s^{-1} \tau(z, Z_i), \quad s \in G, \quad z \in P, \quad Z_i \in T_z^P. \end{array} \right.$$

This is tantamount to requiring that

$$(2.1a) \quad r(s) \tau = \tau \quad \text{for all } s \in G$$

(cf. 1.33), or else, if G is connected (12)

$$(2.1b) \quad \theta(u) \tau = 0 \quad \text{for all } u \in L.$$

Note that, since Ad acts trivially on the zero degree part $\Lambda_0^*(P, S(L)) = \Lambda^*(P, \mathbb{R})$, condition (2.1) restricted to the latter simply means *invariance* of the real valued form $\alpha \in \Lambda^*(P, \mathbb{R})$:

$$(2.1a) \quad \left\{ \begin{array}{l} R_s^* \alpha = \alpha \quad \text{for all } s \in G \\ \text{i.e. } \alpha(zs, R_{g^* z} Z_i) = \alpha(z, Z_i), \quad z \in P, \quad Z_i \in T_z^P, \quad s \in G. \end{array} \right.$$

We denote by A^* (resp. $A^p, A_k^*, A_k^p, p, k \in \mathbb{N}$) the respective subsets of Ad -equivariant forms in $\Lambda^*(P, S(L))$ (resp. in $\Lambda^p(P, S(L)), \Lambda^*(P, S_k(L)), \Lambda^p(P, S_k(L))$).

Since the $r(s), s \in G$, are zero-grade automorphisms of the GCDA $(\Lambda^*(P, L), d, \times)$ commuting with all $\rho(\Psi), \Psi \in \mathcal{G}$, and leaving the degree k invariant, we have that A^* decomposes as

$$(2.1) \quad A^* = \bigoplus_p A^p = \bigoplus_k A_k^* = \bigoplus_{p,k} A_k^p$$

and is a sub-GCDA of $(\Lambda^*(P, S(L)), d, \times)$ stable under the action ρ of \mathcal{G} thus also of \mathcal{L} (as well as every component of Λ_k^p) and containing A_0^* as a sub-GCDA isomorphic to the GCDA $(\Lambda^*(P, \mathbb{R}), d, \cdot)$.

We note that Ad -invariance allows to simplify the explicit expression of $\rho(\Psi)$. First note that, due to (1.35), $\rho(\Psi)\tau$ is given for $\tau \in \Lambda^p(P, S(L))$ by the same algorithm as for a real valued differential from (cf. 1.13a):

$$(2.3) \quad (\rho(\Psi)\lambda)(z, Z_i) = \lambda(\Psi^{-1}(z), (\Psi^{-1})_{*z} Z_i), \quad z \in P, \quad Z_i \in T_z^P$$

with $(\Psi^{-1})_{*z}$ given by (1.20) for $\Psi = e^{t\Omega}, \Omega \in \mathcal{L}$. Plugging (1.20) in (2.3) and

(12) Generally (2.1b) amounts to requiring (2.1a) for all $s \in G$ within the connected component of the identity.

using the Ad-equivariance property (2.1) now yields the following explicit form of $\rho(\Omega)$ on A^* :

$$(2.4) \quad \left\{ \begin{array}{l} \langle \rho(e^{t\Omega})\tau \rangle(z, Z_i) = \\ = \text{Ad } e^{t\Omega(z)} \langle \tau(z, Z_i - t(L_z)_* e \text{Ad } e^{-t\Omega(z)}(d\Omega(z, Z))) \rangle \\ \tau \in A^*, \quad \Omega \in \mathcal{L}, \quad z \in P, \quad Z_i \in T_z^P \end{array} \right.$$

leading to

$$(2.5) \quad \left\{ \begin{array}{l} \langle \rho(\Omega)\tau \rangle(z, Z_1, \dots, Z_p) = [\Omega(z), \tau(z, Z_1, \dots, Z_p)] - \\ - \sum_{i=1}^p \tau(z, Z_1, \dots, Z_{i-1}, (L_z)_* e \text{Ad } e^{-t\Omega(z)}(d\Omega(z, Z_i), Z_{i+1}, \dots, Z_p)) \\ \tau \in A^*, \quad \Omega \in \mathcal{L}, \quad z \in P, \quad Z_i \in T_z^P \end{array} \right.$$

(for $\tau \in A_0$ one should set $\text{Ad } e^{t\Omega} = \text{id}$ in (2.4) and omit the first term r.h.s of (2.5)). Note that (2.4) reduces to

$$(2.4b) \quad \left\{ \begin{array}{l} \langle \rho(e^{t\Omega})\tau \rangle(z, Z_i) = \text{Ad } e^{t\Omega(z)}(\tau(z, Z_i)) \\ \tau \in hA^*, \quad \Omega \in \mathcal{L}, \quad z \in P, \quad Z_i \in T_z^P \end{array} \right.$$

on the subset $hA^* \subset A^*$ of *horizontal Ad-equivariant differential forms*, singled out as

$$(2.6) \quad \left\{ \begin{array}{l} hA^* = \bigoplus_{p,k} hA_k^p \\ hA_k^p = \langle \tau \in A_k^p; i(u)\tau = 0 \text{ for all } u \in L \rangle \end{array} \right.$$

(equivalently, the Ad-equivariant $\tau \in A^*$ is horizontal whenever it vanishes as soon as one of its arguments is *vertical* i.e. tangent to the fiber). hA^* is a *graded commutative subalgebra of A^* preserved by the action of \mathcal{G}* as follows from (2.4b). Warning: A^* is not a sub-GCDA of A^* , since *not stable under the exterior derivative d* . However

[2.2]. *The set hA_0^* of real valued horizontal G -invariant differential forms is a sub-GCDA of (A_0^*, d, \cdot) stable under the action of \mathcal{G} and isomorphic as a GCDA to the real-valued De Rham complex $\Lambda^*(M, \mathbb{R}), d, \cdot$ on the base.*

We recall that the isomorphism is given as follows (13)

$$(2.7) \quad \left\{ \begin{array}{l} h \in A_0^P \ni \alpha \leftrightarrow \alpha' \in \Lambda^P(M, \mathbb{R}) \\ \alpha' \rightarrow \alpha = \pi^* \alpha' \\ \alpha \rightarrow \alpha' : \alpha'(x, X_i) = \alpha(z, Z_i) \\ \quad \quad \quad (\text{any } z \in P \text{ and } Z_i \in T_z^P \text{ with } \pi(z) = x, \pi_{*z} Z = X_i) \end{array} \right.$$

The fact that hA_0 is a sub-GCDA of A_0 is seen as follows: remembering that (θ, i) is an action of the Lie algebra L on $\Lambda^*(P, \mathbb{R})$, $A_0 = \bigcap_{u \in L} \ker u$ is closed for the wedge product due to the derivation property of $i(u)$, $u \in L$; and closed for d , because, for $\alpha \in A^0$ (cf. Appendix A, (A.14)) $i(u)d\alpha = -di(u)\alpha + \theta(u)\alpha = 0$.

[2.3]. The DGL $(A_1^*, d, [\cdot])$ of Ad-equivariant L-valued differential forms

Since the $r(s), s \in G$, are zero-grade automorphisms of the DGL $(\Lambda^*(P, L), d, [\cdot])$ commuting with all $\rho(\Psi)$, $\Psi \in \mathcal{G}$, we have that A_1 is a sub-DGL of $(\Lambda^*(P, L), d, [\cdot])$ stable under the action ρ of \mathcal{G} (thus also of \mathcal{L}) (14). The importance of A_1 lies in the fact that it contains as subset both \mathcal{L} and the set \mathbf{A} of connection one-forms on P . Our identification of \mathcal{L} with the smooth Ad-equivariant functions on P (cf. (1.5)) is now expressed as

$$(2.8) \quad \mathcal{L} = A_1^0.$$

We recall, on the other hand, that the set \mathbf{A} of connection one-forms on P is the subset of $a \in A_1^1$ singled out by the following specification of the value of a on vertical vectors:

$$(2.9) \quad a(z, Z) = L_z^{-1} Z, \quad z \in P, \quad Z \in T_z^P \quad \text{with} \quad \pi_{*z} Z = 0$$

alternatively

$$(2.9a) \quad a(z, \theta(u)) = u, \quad u \in L.$$

We recall that \mathbf{A} is an affine subspace of A_1^1 modelled on hA_1^1 (15). The curvature

(13) One has $d\alpha \leftrightarrow d\alpha'$ and $\alpha_1 \wedge \alpha_2 \leftrightarrow \alpha'_1 \wedge \alpha'_2$ because the pull back π^* commutes with d and with the wedge product.

(14) Note however that the replacement of (A.14) by (1.37) prevent $hA_k^* = (\tau \in A_k^*; i(u)\tau = 0 \text{ for all } u \in L)$ – hence in particular hA_1^* – to be stable under d : hA_1^* is merely closed for the product $[\cdot]$ (and hA^* for the product \times).

(15) i.e. $a, a' \in \mathbf{A}, \lambda, \lambda' \in \mathbb{R}, \lambda + \lambda' = 1$ imply $\lambda a + \lambda' a' = \mathbf{A}$; and $a' - a \in \mathbf{A}$.

of a is

$$(2.10) \quad F^a = da + \frac{1}{2} [a \wedge a],$$

it is an element of A_1^2 . The exterior covariant derivation determined by a is the map $D^a : A_1^* \rightarrow A_1^*$ given by (16)

$$(2.11) \quad D^a \lambda = d\lambda + [a, \lambda], \quad \lambda \in A_1^*.$$

We state for further reference the expression of (2.5) for $\tau = a \in \mathbf{A}$ and $\tau = \Omega' \in \mathcal{L}$: we have

$$(2.12) \quad \rho(\Omega)a = -[a \wedge \Omega] - d\Omega, \quad \Omega \in \mathcal{L}, \quad a \in \mathbf{A}$$

(where we used (2.9)) and

$$(2.13) \quad \rho(\Omega)\Omega' = [\Omega \wedge \Omega'], \quad \Omega, \Omega' \in \mathcal{L}.$$

We end up this section with a remark on the «descent» by means of the $P \in S^*(L)$ (cf. [1,4]): using (1.47) we see that, if P is Ad-equivariant, i.e. if $P \cdot \text{Ad } s = P, s \in G, P$ commutes with $r(s), s \in G$: for $P \in I_k(S)$ we have

$$(2.14) \quad P \circ r(s) = r(s) \circ P$$

hence P leaves A^* invariant.

§3. THE COHOMOLOGY ALGEBRA (Φ^*, δ, \wedge) OF \mathcal{L}

Constructing the cohomology algebra of a Lie algebra is a standard procedure which can be applied to any Lie algebra over an arbitrary abelian ring. We now describe this construction in the case where we need it: that of the Lie algebra \mathcal{L} , taken a Lie algebra over $\mathcal{C}^\infty(M)$. We noted in [1.1] that $\mathcal{L} = A_1^0$ is a Lie algebra over $\mathcal{C}^\infty(M)$ acting by pointwise multiplication – abstractly the multiplication of elements of A_1^0 by those of A_0^0 , with the following identification of $\mathcal{C}^\infty(M)$ with A_0^0 :

$$(3.1) \quad \mathcal{C}^\infty(M) \ni a = a' \in A_0^0 \quad \text{iff} \quad \begin{cases} a'(z) = a(\pi z), & z \in P \\ \text{i.e. } a'(x) = a(z), & z \in P, \quad \pi z = x \end{cases}$$

We now consider the direct sum

(16) Unlike $d, D^a, a \in \alpha$, leaves $h\Lambda_1$ stable.

$$(3.2) \quad \Phi^* = \bigoplus_{\alpha \in \mathbb{N}} \Phi^\alpha$$

where Φ^α is the set of alternate $\mathcal{C}^\infty(M)$ -valued α -forms φ (17) on \mathcal{L} ($\Phi^0 = \mathcal{C}^\infty(M)$) local in the following sense: the forms φ are of the type:

$$(3.5) \quad \varphi : (\Omega_1, \dots, \Omega_\alpha) \in \mathcal{L} \times \dots (\alpha \text{ times}) \dots \times \mathcal{L} \rightarrow \theta(D_1 \Omega_1, \dots, D_\alpha \Omega_\alpha) \in \mathcal{C}^\infty(M)$$

with θ a $\mathcal{C}^\infty(M)$ -valued $\mathcal{C}^\infty(M)$ -linear α -forms on \mathcal{L} and the $D_i, i = 1, \dots, \alpha$, linear differential operators: $\mathcal{L} \rightarrow \mathcal{L}$. Φ^* becomes a GCDA (Φ^*, δ, \wedge) if equipped with the *wedge product*

$$(3.6) \quad \left\{ \begin{array}{l} (\varphi \wedge \psi)(\Omega_\alpha, \dots, \Omega_{\alpha+\beta}) = \\ = \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} \chi(\sigma) \varphi(\Omega_{\sigma_1}, \dots, \Omega_{\sigma_\alpha}) \psi(\Omega_{\sigma(\alpha+1)}, \dots, \Omega_{\sigma(\alpha+\beta)}) \\ \Omega_1, \dots, \Omega_{\alpha+\beta} \in \mathcal{L} \end{array} \right.$$

and the *coboundary operator* (of Lie algebra cohomology):

$$(3.7) \quad \left\{ \begin{array}{l} \delta\varphi(\Omega_0, \dots, \Omega_\alpha) = \sum_{0 \leq i < j < \alpha} (-1)^{i+j} \varphi([\Omega_i, \Omega_j], \Omega_0, \dots, \hat{\Omega}_i, \dots, \hat{\Omega}_j, \dots, \Omega_\alpha) \\ \Omega_0, \dots, \Omega_p \in \mathcal{L} \\ \delta = 0 \quad \text{on} \quad \Phi^0. \end{array} \right.$$

Explicitly, δ is a linear operator of grade 1 fulfilling

$$(3.8) \quad \delta^2 = 0$$

$$(3.9) \quad \delta(\varphi \wedge \psi) = (\delta\varphi) \wedge \psi + (-1)^\alpha \varphi \wedge \delta\psi, \quad \varphi \in \Phi^\alpha, \quad \psi \in \Phi^*.$$

For the proof of these classical facts we refer to, e.g. [1, Corollary 9] (case $A = V = \mathcal{C}^\infty(M)$).

§4. COHOMOLOGY OF \mathcal{L} RELATIVE TO THE REPRESENTATION SPACES Λ_k^* . THE DOUBLE COMPLEXES (Δ_k^{}, d, s) (\mathcal{A}_k^{**}, D, s). THE GCDA's $(*\Delta, \Delta, \mathbf{x})$, $(*\mathcal{A}_0, \Delta, \mathbf{x})$, $(*\mathcal{A}, \Delta, \mathbf{x})$, $(*\mathcal{A}_0, \Delta, \mathbf{x})$. THE DGLs $(*\mathcal{A}_1, \Delta, [\wedge])$, $(*\mathcal{A}_1, \Delta, [\wedge])$**

Having at hand all prerequisites, we now come to our subject proper, a combi-

(17) The subset of those φ obtained with operators D_i of degree zero is a sub-GCDA (Φ_0^* , δ, \wedge) of (Φ^*, δ, \wedge) .

nation of the structures in sections 2 and 3. The matching is obtained by a canonical construction available whenever one has a representation ρ of a Lie algebra (over some abelian ring) on a module over this ring. In our case the Lie algebra is \mathcal{L} , the ring $\mathcal{C}^\infty(M)$, the $\mathcal{C}^\infty(M)$ -module Λ^* ; and the representation is ρ (direct sum of the restrictions of ρ to the component Λ_k^p). The additional structure of Λ^* , resp. Λ_0^* as GCDAs (and of Λ_1^* as a DGL) produces interesting extra features: products \star and $[\cdot, \cdot]$, a double complex and associated total complex, GCDA and GDL structures, the BRS transformations, etc. We now describe the «Chevalley cohomology» which is «local» by construction.

From here on we use the shorthands $\Lambda^* = \Lambda^*(P, S(L))$, $\Lambda^p = \Lambda^p(P, S(L))$, $\Lambda_k^p = \Lambda^p(P, S_k(L))$ (in particular $\Lambda_0^p = \Lambda^p(P, \mathbb{R})$, $\Lambda_1^p = \Lambda^p(P, L)$); and the corresponding shorthands for Λ replaced by A .

[4.1]. The double complex $(\Lambda^{**}, d, \delta)$

Consider the tensor product over $\mathcal{C}^\infty(M) = A_0^0$ ⁽¹⁸⁾

$$(4.1) \quad \Lambda^{**} = \Lambda^* \otimes \Phi^*,$$

doubly graded as the direct sum of subspaces

$$(4.2) \quad \Lambda^{p\alpha} = \Lambda^p \otimes \Phi^\alpha, \quad a, p \in \mathbb{N}$$

themselves splitting into subspaces

$$(4.3) \quad \Lambda_k^{p\alpha} = \Lambda_k^p \otimes \Phi^\alpha \quad (\text{hence } \Lambda^{**} = \bigoplus_k (\Lambda_k^{**} = \bigoplus_{p,\alpha} \Lambda_k^p \otimes \Phi^*))$$

(α is the «ghost number», p the *degree of form*, k the *tensor type*). The elements of Λ^{**} are interpreted as Λ^* -valued multilinear forms on \mathcal{L} ⁽¹⁹⁾, according to the identification

$$(4.4) \quad U = \tau \otimes \varphi \leftrightarrow \begin{cases} U(\Omega_1, \dots, \Omega_\alpha) = \varphi(\Omega_1, \dots, \Omega_\alpha) \tau \\ \text{for all } \Omega_1, \dots, \Omega_\alpha \in \mathcal{L} \end{cases}, \quad \begin{cases} U \in \Lambda^{**} \\ \varphi \in \Phi^\alpha \\ \tau \in \Lambda^* \end{cases}$$

We now turn Λ^{**} into a double complex with *horizontal differential* d and

⁽¹⁸⁾ All the subsequent tensor products are over $\mathcal{C}^\infty(M)$. It is tempting at this point to work with tensorproducts over $\mathcal{C}^\infty(M)$. However, one can, alternatively consider the purely algebraic theory where tensor products are over \mathbb{C} . The subsequent results are then maintained, with appropriate replacements of $\mathcal{C}^\infty(M)$ by \mathbb{C} . On the other hand the present construction can be made replacing ϕ^* by ϕ_0^* (see footnote 16)) with maintenance of all results.

⁽¹⁹⁾ In other terms Λ^{**} is identified with the Λ^* -valued Grassmann space over the algebraic dual L^* of L .

vertical differential s :

$$(4.5) \quad \begin{array}{c} \alpha \text{ ghost number} \\ \uparrow \\ s \uparrow \\ \downarrow d \\ \rightarrow p \text{ degree of form} \end{array}$$

This arises by setting

$$(4.6) \quad (dU)(\Omega_1, \dots, \Omega_\alpha) = d(U(\Omega_1, \dots, \Omega_\alpha)), \begin{cases} U \in \Lambda^{*\alpha} \\ \Omega_1, \dots, \Omega_p \in \mathcal{L} \end{cases}$$

and on the other hand (20)

$$(4.7) \quad \begin{cases} (sU)(\Omega_0, \dots, \Omega_\alpha) = -(-1)^p \sum_{i=0}^p (-1)^i \rho(\Omega_i) U(\Omega_0, \dots, \Omega_i, \dots, \Omega_\alpha) \\ + \sum_{0 < i < j < \alpha} (-1)^{i+j} U([\Omega_i, \Omega_j], \Omega_0, \dots, \Omega_i, \dots, \Omega_j, \dots, \Omega_\alpha) \\ U \in \Lambda^{*\alpha}, \quad \alpha \geq 1; \quad \Omega_0, \dots, \Omega_\alpha \in \mathcal{L} \\ (s\lambda)(\Omega_0) = -\rho(\Omega_0), \quad \lambda \in \Lambda^{*0} \end{cases}$$

(in fact $sU = -(-1)^p \delta_p U$, with δ_p the coboundary operator of the cohomology of \mathcal{L} relative to the representation space Λ^*). With these definitions Λ^{**} (in fact each Λ_k^{**} , $k \in \mathbb{N}$, acquires the structure of a double complex i.e., we have

$$(4.8) \quad \begin{cases} d^2 = 0 \\ s^2 = 0 \\ sd + ds = 0 \end{cases}$$

The corresponding total complex $(*\Lambda, \Delta)$ is defined as follows: the (single) total grading is

(20) The minus sign r.h.s. of (4.1) is to ensure the traditional minus signs r.h.s. of the B.R.S. relations cf. Section 5 below.

$$(4.9) \quad \left\{ \begin{array}{l} * \Lambda (= \Lambda^{**}) = \bigoplus_n^n \Lambda \\ \text{with } {}^n \Lambda = \bigoplus_{\alpha+p=n} \Lambda^{p\alpha} \end{array} \right.$$

and the *total derivative* is

$$(4.10) \quad \Delta = d + s$$

fulfilling

$$(4.8a) \quad \Delta^2 = 0$$

(note that d, s and Δ are of grade 1 for the total grading). Proof of these facts: (4.8): $d^2 = d^2 \otimes \text{id}_{\Phi^*} = 0$. Last line (4.8): the coboundary operator δ_ρ commutes with d since acting «internally» (on the arguments of U) whilst d acts «externally» (on the value of U) – the factor $-(-1)^p$ then turns commutation into anticommutation. Second line (4.8): s is known to be a coboundary operator: for a proof we refer to e.g. [1, Corollary 9].

[4.2]. The GCDAs $(\Lambda^*, \Delta, \times)$ and $(\Lambda_0^*, \Delta, \times)$

We now introduce a bilinear product \times on Λ^{**} . We set

$$(4.11) \quad (\tau \otimes \varphi) \times (\tau' \otimes \varphi') = (-1)^{\alpha\beta'} \cdot (\tau \times \tau') \otimes (\varphi \wedge \varphi'), \quad \left\{ \begin{array}{l} \tau \in \Lambda^*, \tau' \in \Lambda^{\beta'} \\ \varphi \in \Lambda^\alpha, \varphi' \in \Phi^* \end{array} \right.$$

alternatively

$$(4.11a) \quad \left\{ \begin{array}{l} (U \times V)(\Omega_1, \dots, \Omega_{\alpha+\beta}) = \\ = (-1)^{\alpha\beta} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} \chi(\sigma) U(\Omega_{\sigma_1}, \dots, \Omega_{\sigma_\alpha}) \times V(\Omega_{\sigma_{\alpha+1}}, \dots, \Omega_{\sigma_{\alpha+\beta}}). \\ U \in \Lambda^{**}, \quad V \in \Lambda^{q\beta}, \quad \Omega_1, \dots, \Omega_{\alpha+\beta} \in \mathcal{L}. \end{array} \right.$$

We now have that $(*\Lambda, \Delta, \times)$ is a GCDA with $(*\Lambda_0, \Delta, \times)$ as a sub-GCDA ⁽²¹⁾. This is proven as follows: the fact that $(*\Lambda, \times)$ is a graded-commutative algebra is proven in Appendix A, cf. (A.30), (A.31). We already proved that Δ is of grade 1 and square zero. Now d and s , and hence Δ , are derivatives for the product \times .

⁽²¹⁾ $*\Lambda_0$ is Λ_0^{**} equipped with the total grading (4.9).

We check that for d : given $U \in \Lambda^{p\alpha}$, $V \in \Lambda^{q\beta}$, $\Omega_{\alpha^1}, \dots, \Omega_{\alpha+\beta} \in \mathcal{L}$ we have, from (4.6), since d is a graded derivation for \times :

$$\begin{aligned}
 (4.12) \quad d(U \times V)(\Omega_1, \dots, \Omega_{\alpha+\beta}) &= \\
 &= (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} x(\sigma) \cdot \\
 &\quad \cdot (dU(\Omega_{\sigma_1}, \dots, \Omega_{\sigma_\alpha}) \times (V(\Omega_{\sigma(\alpha+1)}, \dots, \Omega_{\sigma(\alpha+\beta)} + \\
 &\quad + (-1)^p U(\Omega_{\sigma_1}, \dots, \Omega_{\sigma_\alpha}) \times dV(\Omega_{\sigma(\alpha+1)}, \dots, \Omega_{\sigma(\alpha+\beta)})) = \\
 &= (-1)^{\alpha q} (-1)^{\alpha q} (dU \times V)(\Omega_1, \dots, \Omega_{\alpha+\beta}) + (-1)^{p+\alpha(q+1)} (U \times dV)(\Omega_1, \dots, \Omega_{\alpha+\beta})
 \end{aligned}$$

The proof for s is obtained as follows invoking Theorem [1.8] (vi) of [1]: denoting by ∇ the product on Λ^{**} obtained by discarding the factor $(-1)^{\alpha q}$ r.h.s. of definition (4.12a), the fact that $\rho(\Omega)$, $\Omega \in \mathcal{L}$, is a derivation of (Λ^*, \times) implies that δ_ρ is a derivation for the product ∇ . We then have, for $U \in \Lambda^{p\alpha}$, $V \in \Lambda^{q\beta}$

$$\begin{aligned}
 (4.13) \quad s(U \times V) &= -(-1)^{p+q+\alpha q} \delta_\rho(U \nabla V) = \\
 &= -(-1)^{p+q+\alpha q} (\delta_\rho U) \nabla V + (-1)^\alpha U \nabla \delta_\rho V = \\
 &= -(-1)^{(\alpha+1)q} (sU) \nabla V + (-1)^{p+\alpha+\alpha q} U \nabla sV = \\
 &= sU \times V + (-1)^{p+\alpha} U \times sV.
 \end{aligned}$$

Note that, since d and s preserve the tensor type, *each* (Λ_k^{**}, d, s) is a double complex with total complex $(*\Lambda_k, \Delta)$. Moreover $(*\Lambda_0, \Delta, \times)$ is a sub-GCDA of $(*\Lambda, \Delta, \times)$. The fact that $*\Lambda_0$ is closed for the product \times stems from the fact that Λ_0^* is closed for \times . And $*\Lambda_0$ is stable under Δ (in fact under d and s) because the latter preserve the tensor type.

[4.3]. The DGL $(*\Lambda_1, \Delta, [\wedge])$

The case $k = 1$ deserves special attention. We define a bilinear product on Λ_1^{**} be setting

$$(4.14) \quad [\lambda \otimes \varphi \wedge \mu \otimes \psi] = (-1)^{\alpha q} [\lambda \wedge \mu] \otimes (\varphi \wedge \psi) \left\{ \begin{array}{l} \lambda \in \Lambda^*, \mu \in \Lambda^q \\ \varphi \in \Phi^\alpha, \psi \in \Phi^* \end{array} \right.$$

equivalently

$$(4.14a) \quad \left\{ \begin{array}{l} [A \frown B](\Omega_1, \dots, \Omega_{\alpha+\beta}) = (-1)^{\alpha q} \frac{1}{\alpha!} \frac{1}{\beta!} \sum_{\sigma \in \Sigma_{\alpha+\beta}} x(\sigma) \\ [A(\Omega_{\sigma 1}, \dots, \Omega_{\sigma \alpha}) \frown B(\Omega_{\sigma(\alpha+1)}, \dots, \Omega_{\sigma(\alpha+\beta)})] \\ A \in \mathbb{A}_1^{*\alpha}, B \in \mathbb{A}^{q\beta}, \Omega_1, \dots, \Omega_{\alpha+\beta} \in \mathcal{L}. \end{array} \right.$$

Then $(*\mathbb{A}_1, \Delta, [\frown])$ becomes a DGL (22).

Proof. The fact that $(*\mathbb{A}_1, [\frown])$ is a graded Lie algebra is shown in Appendix A (iv) cf. (A34). Δ preserves \mathbb{A}_1^* and is of square zero and total grade 1. On the other hand d and s , and thus Δ , acts as derivations of $*\mathbb{A}_1$ for the product $[\frown]$. For d this follows from [A.2] (ii) in Appendix A. The proof for s is again obtained by invoking Theorem [1.8] (vi) of [1]: denoting by \square the product on \mathbb{A} obtained by discarding the factor $(-1)^{\alpha q}$ r.h.s. of definition (4.14a), the fact that $\rho(\Omega)$, $\Omega \in \mathcal{L}$ is a derivation of $(\mathbb{A}_1^*, [\frown])$ implies that δ_ρ is a derivation of \mathbb{A}_1^* . We then have, for $A \in \mathbb{A}_1^{p\alpha}, B \in \mathbb{A}^{q\beta}$,

$$(4.15) \quad \begin{aligned} s[A \frown B] &= -(-1)^{p+q+\alpha q} \delta_\rho(A \square B) = \\ &= -(-1)^{p+q+\alpha q} (\delta_\rho A \square B + (-1)^\alpha A \square \delta_\rho B) = \\ &= -(-1)^{(\alpha+1)q} (sA) \square B + (-1)^{p+\alpha+\alpha q} A \square sB = \\ &= [sA \frown B] + (-1)^{p+\alpha} [A \frown sB]. \end{aligned}$$

[4.4]. Action r of G on $\mathbb{A}^{} = *\mathbb{A}$. The operators $\theta(u)$ and $i(u)$, $u \in L$, on \mathbb{A}^{**} . Commutation and derivation properties**

On $\mathbb{A}^{**} = \Lambda^* \otimes \Phi^* (= *\mathbb{A})$ we define the action r of the group G , and the operators $\theta(u)$, $r(u)$, $u \in L$, by tensoring with id_{Φ^*} :

$$(4.16) \quad r(g) = r(g) \otimes \text{id}_{\Phi^*}, \quad g \in G$$

$$(4.17) \quad \theta(u) = \theta(u) \otimes \text{id}_{\Phi^*}, \quad u \in L$$

$$(4.18) \quad i(u) = i(u) \otimes \text{id}_{\Phi^*}, \quad u \in L$$

alternatively, for $U \in \Lambda^{**}$, $\Omega_1, \dots, \Omega_\alpha \in \mathcal{L}$

$$(4.16a) \quad \{r(g)U\}(\Omega_1, \dots, \Omega_\alpha) = r(g)\{U(\Omega_1, \dots, \Omega_\alpha)\}$$

$$(4.17a) \quad \{\theta(u)U\}(\Omega_1, \dots, \Omega_\alpha) = \theta(u)\{U(\Omega_1, \dots, \Omega_\alpha)\}$$

(22) $*\mathbb{A}_1$ is \mathbb{A}_1^{**} equipped with the total grading (4.9).

$$(4.18a) \quad \{i(u)U\}(\Omega_1, \dots, \Omega_\alpha) = i(u)\{U(\Omega_1, \dots, \Omega_\alpha)\}$$

(note that $r(g)$, $g \in G$, preserves the ghost number, the degree of the form, and the tensor type).

We then have the following commutation rules

$$(4.19) \quad i(u)s + s i(u) = 0, \quad u \in L$$

$$(4.20) \quad \theta(u)s - s \theta(u) = 0, \quad u \in L$$

$$(4.21) \quad r(g)s - s r(g) = 0, \quad g \in G$$

$$(4.22) \quad r(g)d - d r(g) = 0, \quad g \in G$$

$$(4.23) \quad r(g)\Delta - \Delta r(g) = 0, \quad g \in G$$

$$(4.24) \quad i(u)i(v) + i(v)i(u) = 0, \quad u, v \in L$$

$$(4.25) \quad \theta(u)\theta(v) - \theta(v)\theta(u) = \theta([u, v]), \quad u, v \in L$$

$$(4.26) \quad \theta(u)i(v) - i(v)\theta(u) = i([u, v]), \quad u, v \in L$$

$$(4.27) \quad \theta(u)d - d \theta(u) = 0, \quad u \in L$$

$$(4.28) \quad \theta(u)i(u) - i(u)\theta(u) = 0, \quad u \in L$$

$$(4.29) \quad i(u)d + d i(u) = \text{id}_{S(L)} \otimes \theta(u) \otimes \text{id}_{\Phi^*}$$

$$i(u)\Delta + \Delta i(u) = \theta(u) - \text{Ad } u \otimes \text{id}_{\Lambda_0} \otimes \text{id}_{\Phi^*}.$$

Moreover, we have

$$(4.30) \quad \left. \frac{d}{dt} \right|_{t=0} r(e^{tu}) = \theta(\xi_u), \quad u \in L,$$

and the following invariance, resp. derivation properties in $(*\Lambda, \star)$:

$$(4.31) \quad r(g)\{U \star V\} = \{r(g)U\} \star \{r(g)V\}, \quad U, V \in *\Lambda, \quad g \in s$$

$$(4.32) \quad \theta(u)\{U \star V\} = \{\theta(u)U\} \star V + U \star \theta(u)V, \quad U, V \in *\Lambda, \quad u \in L,$$

$$(4.33) \quad i(u)\{U \star V\} = \{i(u)U\} \star V + (-1)^n U \star i(u)V, \quad U \in {}^n\Lambda, \quad V \in *\Lambda, \quad u \in L$$

and in $(*\Lambda_1, [\cdot])$:

$$(4.34) \quad r(g)[A \wedge B] = [r(g)A \wedge r(g)B], \quad A, B \in *\Lambda, \quad g \in G$$

$$(4.35) \quad \theta(u)[A \wedge B] = [\theta(u)A \wedge B] + [A \wedge \theta(u)B], \quad A, B \in *\Lambda, \quad u \in L$$

$$(4.36) \quad i(u)[A \wedge B] = [i(u)A \wedge B] + (-1)^n [A \wedge i(u)B], \quad A \in {}^n\Lambda, \quad B \in *\Lambda, \quad n \in L.$$

Properties (4.19), (4.20) and (4.21) are due to the fact that s acts «internally» whilst $i(u)$, $\theta(u)$ and $r(g)$ act «externally» (23). Property (4.22) through (4.29) follow from the corresponding properties in $^*\Lambda$, resp. $^*\Lambda_1$ via tensoring with $\text{id}_{\mathfrak{q}^*}$ (immediate from [1.3] (ii), (iii)). We check the derivation properties (in particular the fact that properties (4.33), (4.35) hold w.r.t. the total grading). With $U = \tau \otimes \varphi \in \Lambda^p \otimes \Phi^\alpha$ $U' = \tau' \otimes \varphi' \in \Lambda^{p'} \otimes \Phi^{\alpha'}$, we have $U \times U' = (-1)^{\alpha p'} (\tau \times \tau') \otimes (\varphi \wedge \varphi')$, hence

$$(4.37) \quad \begin{aligned} \theta(u)(U \times U') &= (-1)^{\alpha p'} \{(\theta(u)\tau) \times \tau' + \tau \times \theta(u)\tau'\} \otimes (\varphi \wedge \varphi') = \\ &= \{(\theta(u)\tau) \otimes \varphi\} \times (\tau' \otimes \varphi') + (\tau \otimes \varphi) \times \{(\theta(u)\tau') \otimes \varphi'\} = \\ &= \{\theta(u)U \times U'\} + U \times \theta(u)U' \end{aligned}$$

and

$$(4.38) \quad \begin{aligned} i(u)(U \times U') &= (-1)^{\alpha p'} \{i(u)\tau \times \tau' + (-1)^p \tau \times i(u)\tau'\} \otimes (\varphi \wedge \varphi') = \\ &= (-1)^{\alpha p' + \alpha p'} \{i(u)\tau \otimes \varphi'\} \times \{\tau' \otimes \varphi\} + \\ &+ (-1)^{\alpha p' + p + \alpha(p'+1)} (\tau \otimes \varphi) \times \{i(u)\tau' \otimes \varphi'\} = \\ &= i(u)U \times U' + (-1)^{p+\alpha} U \times i(u)U' \end{aligned}$$

and with $A = \lambda \otimes \varphi \in \Lambda_1^p \otimes \Phi^\alpha$, $A' = \lambda' \otimes \varphi' \in \Lambda_1^{p'} \otimes \Phi^{\alpha'}$, thus $A \times A' = (-1)^{\alpha p'} (\lambda \times \lambda') \otimes (\varphi \wedge \varphi')$,

$$(4.39) \quad \begin{aligned} \theta(u)[A \wedge A'] &= (-1)^{\alpha p'} \{[\theta(u)\lambda \wedge \lambda'] + [\lambda \wedge (\theta(u)\lambda')]\} \otimes (\varphi \wedge \varphi') = \\ &= [\theta(u)\lambda \otimes \varphi \wedge \lambda' \otimes \varphi'] + [\lambda \otimes \varphi \wedge \theta(u)\lambda' \otimes \varphi'] = \\ &= [\theta(u)A \wedge A'] + [A \wedge \theta(u)A'] \end{aligned}$$

and

$$(4.40) \quad \begin{aligned} i(u)[A \wedge A'] &= (-1)^{\alpha p'} \{[i(u)\lambda \wedge \lambda'] + (-1)^p [\lambda \wedge i(u)\lambda']\} \otimes (\varphi \wedge \varphi') = \\ &= (-1)^{\alpha p' + \alpha p'} [i(u)\lambda \otimes \varphi \wedge \lambda' \otimes \varphi'] + (-1)^{\alpha p' + p + \alpha(p'+1)} [\lambda \otimes \varphi \wedge i(u)\lambda' \otimes \varphi'] \\ &= [i(u)A \wedge A'] + (-1)^{p+\alpha} [A \wedge i(u)A']. \end{aligned}$$

[4.5]. The Ad-equivariant double complex (Λ_k^{**}, d, s) , GCDAs $(^*\Lambda, \Delta, \times)$ and $(^*\Lambda_0, \Delta, \times)$ and DGL $(^*\Lambda_1, \Delta, [\wedge])$.

With $p, \alpha, k \in \mathbb{N}$ we now consider the fixpoints sets

(23) This implies commutativity of $i(u)$, $\theta(u)$ and $r(g)$ with δ_ρ , turned into anticommutativity for the odd grade $i(u)$ by passing from δ_ρ to s .

$$(4.41) \quad \mathbb{A}_k^{p\alpha} = \{U \in \mathbb{A}_k^{p\alpha}, r(g)U = U \text{ for all } g \in G\} = \\ = A_k^p \otimes \Phi^\alpha$$

equivalently, for G connected,

$$(4.41a) \quad \mathbb{A}_k^{p\alpha} = \{U \in \mathbb{A}_k^{p\alpha}, \theta(u)U = 0 \text{ for all } u \in L\}$$

with the bigraded spaces

$$(4.42) \quad \mathbb{A}_k^{**} = \bigoplus_{p,\alpha} \mathbb{A}_k^{p\alpha}, \quad \mathbb{A}^{**} = \bigoplus_{p,\alpha} \mathbb{A}^{p\alpha}$$

($\mathbb{A}^{p\alpha} = \bigoplus_k \mathbb{A}_k^{p\alpha}$) and the graded spaces

$$(4.43) \quad {}^* \mathbb{A}_k = \bigoplus_n {}^n \mathbb{A}_k, \quad {}^* \mathbb{A} = \bigoplus_n {}^n \mathbb{A}$$

where n is the «total grading»

$$(4.44) \quad {}^n \mathbb{A}_k = \bigoplus_{p+\alpha=n} \mathbb{A}_k^{p\alpha}, \quad {}^n \mathbb{A} = \bigoplus_{p+\alpha=n} \mathbb{A}^{p\alpha}.$$

We now have that $(\mathbb{A}_k^{**}, d, s), k \in \mathbb{N}$ resp. (\mathbb{A}^{**}, d, s) are sub-double complexes of the double complexes $(\mathbb{A}_k^{**}, d, s)$ resp. (\mathbb{A}^{**}, d, s) ; that $({}^* \mathbb{A}, \Delta, \varkappa)$, resp. $({}^* \mathbb{A}_0, \Delta, \varkappa)$ are sub-GCDAs of the GCDAs $({}^* \mathbb{A}, \Delta, \varkappa)$, resp. $({}^* \mathbb{A}_0, \Delta, \varkappa)$; and that $({}^* \mathbb{A}_1, \Delta, [\cdot])$ is a sub-DGL of the DGL $({}^* \mathbb{A}, \Delta, [\cdot])$. These facts immediately result from the commutation of $r(g)$ and d, δ cf. (4.21), (4.22)); the commutation of $r(g)$ and Δ, \varkappa (cf. (4.23), (4.31)); and the additional commutation of $r(g)$ with $[\cdot]$ (cf. (4.34)), $g \in G$. Moreover, with $h \mathbb{A}_0^{**}$ the set of invariant horizontal real differential forms on P :

$$(4.45) \quad \left\{ \begin{array}{l} h \mathbb{A}_0^{**} = \bigoplus_{p,\alpha} h \mathbb{A}_0^{p\alpha} \\ h \mathbb{A}_0^{p\alpha} = \langle F \in \mathbb{A}_0^{p\alpha}; i(u)F = 0 \text{ for all } u \in L \rangle \end{array} \right.$$

and

$$(4.46) \quad {}^n \mathbb{A}_0 = \bigoplus_{p+\alpha=n} \mathbb{A}_0^{p,\alpha}$$

we have that $(h \mathbb{A}_0^{**}, d, s)$ is a sub-bicomplex of $(\mathbb{A}_0^{**}, d, s)$ and $(h^* \mathbb{A}_0, \Delta, \varkappa)$ is a sub-GCDA of $(\mathbb{A}_0, \Delta, \varkappa)$ (this follows from the facts that $h \mathbb{A}_0^{**}$ is stable under both s and d , owing to (4.19) and (4.29) (remember that $\theta(u)$ vanishes on \mathbb{A}^{**}).

Remark. On \mathbb{A}^{**} we have also an action (Θ, I) of the Lie algebra \mathcal{L} , given as follows: for $U \in \mathbb{A}^{**}, \Omega, \Omega_1, \dots, \Omega_j \in \mathcal{L}$:

$$(4.47) \quad \begin{aligned} \{\Theta(\Omega)U\}(\Omega_1, \dots, \Omega_p) &= \\ &= \rho(\Omega)\{U(\Omega_1, \dots, \Omega_p)\} + \sum_{i=1}^{\alpha} U(\Omega_1, \dots, \Omega_{i-1}, [\Omega, \Omega_i], \Omega_{i-1}, \dots, \Omega_p) \end{aligned}$$

$$(4.48) \quad \{I(\Omega)U\}(\Omega_1, \dots, \Omega_{p-1}) = U(\Omega_1, \Omega_1, \dots, \Omega_{p-1}).$$

For a proof of these facts see e.g. [1] Theorem [1.8] (i) (iii).

[4.6]. Alternative definition of $(*\mathbb{A}, \Delta, \times)$

We now describe a different way of looking at the \mathbb{C} -tensor product version of $*\mathbb{A}$ (cf. footnote (18)) which has also some conceptual interest.

Since Λ^* was defined as the tensor product $S(L) \otimes \Lambda_0^*$, the space $*\mathbb{A}$ is the double tensor product ⁽²⁴⁾

$$(4.49) \quad \begin{cases} *\mathbb{A} = S(L) \otimes_{\mathbb{C}} \Lambda_0^* \otimes_{\mathbb{C}} \Phi^* \\ \mathbb{A}_k^{p\alpha} = S_k(L) \otimes_{\mathbb{C}} \Lambda_0^p \otimes_{\mathbb{C}} \Phi^\alpha \end{cases}$$

on which the product \times is defined as follows: we have

$$(4.50) \quad (f \otimes \eta \otimes \varphi) \times (g \otimes \xi \otimes \psi) = (-1)^{\alpha q} (f \smile g) \otimes (\eta \wedge \xi) \otimes (\varphi \frown \psi),$$

$$\begin{cases} f, g \in S(L) \\ \eta \in \Lambda_0^*, \quad \xi \in \Lambda_0^q \\ \varphi \in \Phi^\alpha \quad \psi \in \Phi^*. \end{cases}$$

This can be written alternatively

$$(4.50a) \quad \{f \otimes (\eta \otimes \varphi)\} \times (g \otimes (\xi \otimes \psi)) = (f \smile g) \otimes \{(\eta \otimes \varphi) \wedge (\xi \otimes \psi)\}$$

where the wedge product of the r.h.s. is the skew product of the GCDAs $*\Lambda_0$ and Φ^* , naturally included in $*\mathbb{A}$ as $*\mathbb{A}_0$ (cf. (A.17) in Appendix A). We could thus have constructed $(*\mathbb{A}, \Delta, \times)$ by assembling the algebras $(S(L), \smile)$, (Λ_0^*, d, \wedge) and (Φ^*, δ, \frown) as follows: first build the skew tensor product $(*\mathbb{A}_0, \wedge)$ of graded commutative algebras

$$(4.51) \quad *\mathbb{A}_0 = \Lambda_0^* \otimes \Phi^*,$$

with corresponding grading

⁽²⁴⁾ Written without parentheses according to associativity of tensor product.

$$(4.52) \quad {}^n\Lambda_0 = \bigoplus_{p+\alpha=n} \Lambda_0^p \otimes_{\mathcal{C}^\infty(M)} \Phi^\alpha$$

and skew tensor product (25)

$$(4.53) \quad (\eta \otimes \varphi) \wedge (\xi \otimes \psi) = (-1)^{\alpha q} (\eta \wedge \xi) \otimes (\varphi \wedge \psi) \left\{ \begin{array}{l} \eta \in \Lambda_0^*, \xi \in \Lambda_0^q \\ \varphi \in \Phi^\alpha, \psi \in \Phi^* \end{array} \right.$$

then build the tensor product

$$(4.54) \quad {}^*\Lambda = S(L) \otimes_{\mathbb{R}} {}^*\Lambda_0$$

with product

$$(4.55) \quad (f \otimes S) \times (g \otimes T) = (f \cdot g) \otimes (S \cdot T), \left\{ \begin{array}{l} f, g \in S(L) \\ S, T \in {}^*\Lambda_0 \end{array} \right.$$

We obtain in this way the graded commutative algebra $({}^*\Lambda, \wedge)$ embedded as such in the graded commutative algebra $({}^*\mathcal{A}, \times)$. We recapitulate our operators: they are all totally split tensorially

$$(4.56) \quad r(s) = \text{Ad } s \otimes R_s^* \otimes \text{id}_{\Phi^*}, \quad s \in G$$

$$(4.57) \quad \rho(\Psi) = \text{id}_{S(L)} \otimes \rho(\Psi) \otimes \text{id}_{\Phi^*}, \quad \Psi \in \mathcal{G}$$

$$(4.58) \quad \rho(\Omega) = \text{id}_{S(L)} \otimes \rho(\Omega) \otimes \text{id}_{\Phi^*}, \quad \Omega \in \mathcal{L}$$

$$(4.59) \quad d = \text{id}_{S(L)} \otimes d \otimes \text{id}_{\Phi^*}$$

$$(4.60) \quad s_0 = \text{id}_{S(L)} \otimes (-1)^p \otimes \delta$$

$$(4.61) \quad i(u) = \text{id}_{S(L)} \otimes i(u) \otimes \text{id}_{\Phi^*}, \quad u \in L$$

$$(4.62) \quad \theta(u) = \text{Ad } u \otimes \text{id}_{\Lambda_0^*} \otimes \text{id}_{\Phi^*} + \text{id}_{S(L)} \otimes \theta(u) \otimes \text{id}_{\Phi^*}, \quad u \in L$$

$$(4.63) \quad (26) \quad P = P \otimes \text{id}_{\Lambda_0^*} \otimes \text{id}_{\Phi^*}$$

except for the operator $\rho \wedge$ in

$$(8.64) \quad \Delta = d + s_0 + \rho \wedge$$

(25) Δ in Λ_0^* is not the skew tensor product of derivatives d and δ . The latter is $d + s_0 = \Delta - \rho \wedge$, see below.

(26) See below [4.7] for a study of P on ${}^*\Lambda$.

given by

$$(4.65) \quad \rho^\wedge = \text{id}_{S(L)} \otimes \rho \cdot$$

with

$$(4.66) \quad \begin{aligned} & \{\rho^\wedge(\alpha \otimes \varphi)\}(\Omega_0, \dots, \Omega_\alpha) = \\ & = \sum_{i=1}^{\alpha} (-1)^i \varphi(\Omega_0, \dots, \hat{\Omega}_i, \dots, \Omega_\alpha) \rho(\Omega) \alpha, \end{aligned} \left\{ \begin{array}{l} \alpha \in \Lambda^* \\ \varphi \in \Phi^\alpha \\ \Omega_0, \dots, \Omega_\alpha \in \mathcal{L} \end{array} \right.$$

Note that the product $[\wedge]$ on ${}^*\Lambda_1 = L \otimes \Lambda_0^*$ is given by

$$(4.67) \quad [(u \otimes S) \wedge (v \otimes T)] = [u, v] \otimes (S \wedge T), \left\{ \begin{array}{l} u, v \in L \\ S, T \in {}^*\Lambda_0 \end{array} \right.$$

[4.7]. Covariant derivatives. \mathcal{A} -connections

Appendix B applied to the DGL $({}^*\Lambda_1, \Delta, [\wedge])$ yields a linear assignment, to each $B \in {}^1\mathcal{A}_1$, of his «covariant derivative»

$$(4.68) \quad \mathcal{D}^B = \Delta + [B \wedge \cdot]$$

with square

$$(4.69) \quad (\mathcal{D}^B)^2 = [\mathcal{F}^B \wedge \cdot]$$

where the «curvature» of B , $\mathcal{F}^B \in {}^2\Lambda_1$, given by

$$(4.70) \quad \mathcal{F}^B = \Delta B + \frac{1}{2} [B \wedge B]$$

fulfills a «Bianchi identity»

$$(4.71) \quad \mathcal{D}^B \mathcal{F}^B = 0.$$

Now we have a unique graded derivation (again denoted \mathcal{D}^B) of $({}^*\Lambda, \times)$ restricting to \mathcal{D}^B on ${}^*\Lambda_1$, and to Δ on ${}^*\Lambda_0$ (in fact the latter is the sum of Δ and of the unique graded derivation restricting to $[B \wedge \cdot]$ on ${}^*\Lambda_1$ and to zero on ${}^*\Lambda_0$).

\mathcal{D}^B is given by (27).

$$(4.72) \quad \left\{ \begin{array}{l} \mathcal{D}^B(A_1 \times \dots \times A_k) \\ = \sum_{i=1}^k (-1)^{k-\sum_{j=1}^{i-1} n_j} A_1 \times \dots \times A_{i-1} \times \mathcal{D}^B A_i \times A_{i+1} \times \dots \times A_k \\ A_i \in {}^{n_i} \Lambda_1 \end{array} \right.$$

\mathcal{D}^B thus defined on ${}^* \Lambda$ fulfills (29)

$$(4.73) \quad (\mathcal{D}^B)^2 = [\mathcal{F}^B \wedge \cdot]$$

$$(4.74) \quad \theta(u) \cdot \mathcal{D}^B - \mathcal{D}^B \cdot \theta(u) = [\theta(u) B \wedge \cdot], \quad u \in L$$

$$(4.75) \quad i(u) \circ \mathcal{D}^B + \mathcal{D}^B \cdot i(u) = \theta(u) + [i(u) B \wedge \cdot] - \text{id}_{\Lambda_0^*} \otimes \text{Ad } u \otimes \text{id}_{\mathfrak{g}^*}, \quad u \in L.$$

Proof. (4.73) both sides are derivations, coinciding on ${}^* \Lambda_0$ and ${}^* \Lambda_1$.

(4.74) (resp. (4.75)): both sides are graded derivations (resp. derivations) vanishing on ${}^* \Lambda_0$ and coinciding on ${}^* \Lambda_1$: as results by combining (4.27) with (4.32) rewritten as (4.76) (resp. (4.29) with (4.38) rewritten as (4.77))

$$(4.76) \quad \theta(u) \circ [B \wedge \cdot] - [B \wedge \cdot] \circ \theta(u) = [\theta(u) B \wedge \cdot], \quad B \in {}^1 \Lambda_1$$

$$(4.77) \quad i(u) \circ [B \wedge \cdot] + [B \wedge \cdot] \circ i(u) = [i(u) B \wedge \cdot], \quad u \in L.$$

Let us call **A**-connections the elements of ${}^1 \mathbf{A}_1$ fulfilling

$$(4.78) \quad i(u) A = u \in {}^0 \mathbf{A}_0.$$

Let A be a **A**-connection: we have the commutation rules

$$(4.79) \quad \theta(u) \mathcal{D}^A - \mathcal{D}^A \theta(u) = 0, \quad u \in L$$

$$(4.80) \quad i(u) \mathcal{D}^A + \mathcal{D}^A i(u) = \theta(u), \quad u \in L$$

showing that \mathcal{D}^A preserves ${}^* \mathbf{A}$ and $h_* {}^* \mathbf{A}$. Moreover, in that case \mathcal{F}^A belongs to $h^2 \mathbf{A}_1$ (in fact \mathcal{D}^A preserves ${}^* \mathbf{A}$ for all $A \in {}^* \mathbf{A}_1$).

(27) Generally, for $A \in {}^p \Lambda_1$, p even (resp. p odd), there is a unique derivation (resp. graded derivation) of $({}^* \Lambda, \times)$ restricting to $[A \wedge \cdot]$ on ${}^* \Lambda_1$ and to zero on ${}^* \Lambda_0$.

(28) One has also the same formula with $[B \wedge \cdot]$ instead of \mathcal{D}^B .

(29) With the understanding that $[A \wedge \cdot]$ is the extension described in footnote 7 above.

Proof. If $A \in {}^* \mathbf{A}_1$, $\theta(u)A = 0$, ensuring (4.79), which in turn implies $\mathcal{D}^A {}^* \mathbf{A} \subset \subset {}^* \mathbf{A}$. If in addition $i(u)A = u$ (requiring that $A \in {}^1 \mathbf{A}_1$), we have $[(i(u)A \wedge \cdot)] = [u \wedge \cdot] = \text{id}_{\Lambda_0^*} \otimes \text{Ad } u \otimes \text{id}_{\Phi^*}$, hence (4.75) then reduces to (4.80), which implies $\mathcal{D}^A h^* \mathbf{A} \subset h^* \mathbf{A}$.

We end up this section with a discussion of the descent from ${}^* \mathbf{A}_k$ to ${}^* \mathbf{A}_0$ by means of polynomials $P \in I_k(L)$.

[4.8]. Let P be an *Ad*-equivariant polynomial of degree k on L : $P \in I_k(L)$. Setting

$$(4.81) \quad P(\tau \otimes \varphi) = P(\tau) \otimes \varphi, \quad \tau \in \Lambda^*, \quad \varphi \in \Phi^*,$$

in other terms

$$(4.81a) \quad \{PU\}(\Omega_1, \dots, \Omega_\alpha) = P\{U(\Omega_1, \dots, \Omega_\alpha)\}, \quad U \in \Lambda^{*\alpha}, \quad \Omega_1, \dots, \Omega_\alpha \in \mathcal{L}$$

defines a linear map: $P : {}^* \Delta_k \rightarrow {}^* \Delta_0$ which preserves the ghost number and the degree of form, commutes with the action of G :

$$(4.82) \quad P \circ r(s) = r(s) \circ P, \quad s \in G$$

(hence maps ${}^* \mathbf{A}_k$ into \mathbf{A}_0) and fulfills

$$(4.83) \quad P \circ s = s \circ P$$

$$(4.84) \quad P \circ d = d \circ P$$

$$(4.85) \quad P \circ \Delta = \Delta \circ P, \quad u \in L$$

$$(4.86) \quad P \circ \theta(u) = \theta(u) \circ P, \quad u \in L$$

$$(4.87) \quad P \circ i(u) = i(u) \circ P, \quad u \in L$$

and

$$(4.88) \quad P \circ \mathcal{D}^A = \Delta \circ P, \quad A \in {}^1 \Delta_1$$

$$(4.89) \quad P \circ [A \wedge \cdot] = 0, \quad A \in {}^1 \Delta_1.$$

Proof. (4.83) through (4.87) result at sight from (4.59) through (4.66) (remember that $P \cdot \text{Ad } u = 0$ since $P \in I_k(L)$). (4.88) follows from (4.85) and (4.89) which immediately follows from the

LEMMA. Let $u \in L, f \in S(L), S, T \in \Delta_0$ we have:

$$(4.90) \quad [(u \otimes S) \wedge (f \otimes T)] = \text{Ad } u(f) \otimes (S \wedge T)$$

indeed $P \circ \text{Ad } u = 0$ then implies $P \circ [(u \otimes \Psi) \wedge \cdot] = 0$.

Proof of the Lemma. Denote the r.h.s. of (4.91) by $\delta_{uS}(f \otimes T)$, we want to have $\delta_{uS} = [(u \otimes S) \wedge \cdot]$. This holds on Λ_1 by definition cf. (4.68), and trivially on Λ_0 . To prove that it holds everywhere, it thus suffice to check that δ_{uS} is a derivation of the same type as $[(u \otimes S) \wedge \cdot]$ (30). We have indeed, for $S \in {}^n \Lambda_0$, $T \in {}^n \Lambda_0$

$$\begin{aligned}
 (4.91) \quad & (\delta_{uS}(f \otimes T)) \times (f' \otimes T') + (-1)^{mn}(f \otimes T) \times \delta_{uS}(f' \otimes T') = \\
 & = (\text{Ad } u(f) \otimes (S \wedge T)) \times (f' \otimes T') + (-1)^{mn}(f \otimes T)(\text{Ad } u(f') \otimes (S \wedge T')) \\
 & = (\text{Ad } u(f) - f') \otimes (S \wedge T \wedge T') + (-1)^{mn}(f - \text{Ad } u(f')) \otimes (S \wedge T \wedge T') = \\
 & = (\text{Ad } u(f)] f' + f \text{Ad } u(f'))(S \wedge T \wedge T') = \\
 & = \text{Ad } u(f - f') \otimes S \wedge T \wedge T' = \delta_{uS}(f \otimes T) \times (f' \otimes T'). \quad \blacksquare
 \end{aligned}$$

§5. THE BRS RELATIONS

We now exhibit the BRS relations as realized geometrically within the double complex $(\mathbf{A}_1^{**}, d, s)$ equipped with the product $[\wedge]$.

As we noticed in section 1, the set \mathbf{A} of connection one-forms is contained in \mathbf{A}_1^1 , embedded in \mathbf{A}_1^{**} as $\mathbf{A}_1^{10} = A_1^1 \otimes \mathbb{R}$.

On the other hand, let ω be the «tautological form» on \mathcal{L} (equal to the identity map).

$$(5.1) \quad \omega(\Omega) = \Omega, \quad \Omega \in \mathcal{L}.$$

Remembering that we identified \mathcal{L} with A_1^0 , we set that ω as given by (5.1) appears as an element of $\mathbf{A}_1^{01} = A_1^0 \otimes L$. Hence \mathbf{A}_1^{**} accomodates both a and ω

$$(5.2) \quad \begin{cases} a \in \mathbf{A}_1^{10} \\ \omega \in \mathbf{A}_1^{01}. \end{cases}$$

We therefore may consider $sa, s\omega, d\omega, [a \wedge \omega]$ and $[\omega \wedge \omega]$ in $\mathbf{A}_1^{**} \subset \mathbf{A}^{**}$: in this sense we then have the *BRS relations*

$$(5.3) \quad \begin{cases} sa = -d\omega - [a \wedge \omega] \\ s\omega = -\frac{1}{2} [\omega \wedge \omega]. \end{cases}$$

(30) Cf. (B.8) in Appendix 8.

Proof of these relations by definition (cf. [4.7]), $sa \in \mathbf{A}^{**}$ is given by

$$(5.4) \quad (sa)(\Omega) = \rho(\Omega)a = -d\Omega - [a \wedge \Omega], \quad \Omega \in \mathcal{L}$$

(cf. (2.12)), whilst, also by definition (cf. (4.6a), (5.1)):

$$(5.5) \quad (d\omega)(\Omega) = d(\omega(\Omega)) = d\Omega, \quad \Omega \in \mathcal{L}$$

and (cf. (4.12a))

$$(5.6) \quad [a \wedge \omega](\Omega) = [a \wedge \omega(\Omega)] = [a \wedge \Omega],$$

proving the first line in (4.3). As for the second, we have, by definition of $s\omega \in \mathbf{A}_1^{02}$

$$(5.7) \quad \begin{aligned} s\omega(\Omega_0, \Omega_1) &= -(\rho(\Omega_0)\omega(\Omega_1) - \rho(\Omega_1)\omega(\Omega_0) - \omega([\Omega_0, \Omega_1])) = \\ &= -(\rho(\Omega_0)\Omega_1 - \rho(\Omega_1)\Omega_0 - [\Omega_0, \Omega_1]) = \\ &= -\{[\Omega_0 \wedge \Omega_1] - [\Omega_1 \wedge \Omega_0] - [\Omega_0, \Omega_1]\} = \\ &= -[\Omega_0 \wedge \Omega_1] \end{aligned}$$

(we used (4.7) and (2.13)). On the other hand

$$(5.8) \quad \begin{aligned} [\omega \wedge \omega](\Omega_0, \Omega_1) &= [\omega(\Omega_0) \wedge \omega(\Omega_1)] - [\omega(\Omega_1) \wedge \omega(\Omega_0)] = \\ &= [\Omega_0 \wedge \Omega_1] - [\Omega_1 \wedge \Omega_0] = 2[\Omega_0 \wedge \Omega_1] \end{aligned}$$

proving the second line in (5.3).

Remark. The «ghost» ω arising in the physical literature is an anticommuting «field» $\omega(x)$ with values linear maps $\mathcal{L} \rightarrow L$. Specifically, assuming the principal bundle \mathbf{P} trivial (so that \mathcal{L} consists of smooth maps $\Omega : M \rightarrow L$) and choosing a base e_α in L (with dual base e^α in the dual L^* of L), we have, for $x \in M$

$$(5.9) \quad \omega(x) = \sum_{\alpha} \omega^\alpha(x) e_\alpha$$

i.e.,

$$(5.9a) \quad \langle \omega(x), \Omega \rangle = \sum_{\alpha} \langle \omega^\alpha(x), \Omega \rangle e_\alpha, \quad \Omega \in \mathcal{L}$$

where $\omega^\alpha(x) \in \mathcal{L}^*$ is the dual base of the «base» $\delta_x \otimes e_\alpha$ of $\Omega = \mathcal{C}^\infty(M) \otimes L$ in the following sense

$$(5.10) \quad \langle \omega^\alpha(x), \delta_y \otimes e_\beta \rangle = \delta_\beta^\alpha \delta(x - y), \quad x, y \in M.$$

In the physical literature, the operator s is defined by requiring

$$(5.11) \quad s\omega^\alpha(x) = -\frac{1}{2} f_{\beta\gamma}^\alpha \omega^\beta(x) \otimes \omega^\gamma(x), \quad x \in M$$

where the $f_{\beta\gamma}^\alpha$ are the structure constants of L :

$$(5.12) \quad [e_\beta, e_\gamma] = \sum_\alpha f_{\beta\gamma}^\alpha e_\alpha$$

these relations implying

$$(5.13) \quad s\omega(x) = -\frac{1}{2} [\omega(x), \omega(x)], \quad x \in M,$$

in the sense that one has, for $\Omega_0, \Omega_1 \in \mathcal{L}$:

$$(5.13a) \quad s\omega(x) (\Omega_0, \Omega_1) = \frac{1}{2} [(\omega(x))(\Omega_0), \omega(x)(\Omega_1)].$$

The relation between our $\omega = \text{id}_\varphi$ and the above «ghost» is given as follows: one has

$$(5.14) \quad \{\omega(x)\}(\Omega) = \omega(\Omega)(x) = \Omega(x), \quad x \in M, \quad \Omega \in \mathcal{L}.$$

Indeed, for $\Omega = f \otimes u$, $f \in \mathcal{C}^\infty(M)$, $u \in L$, one has, from (5.9a) and (5.10)

$$\begin{aligned} (5.15) \quad \omega(x) (f \otimes u) &= \sum_\alpha (\omega^\alpha(x))(f \otimes u) e_\alpha \\ &= \sum_\alpha (\omega^\alpha(x)) \left(\int f(y) \delta_y dy \otimes \sum u^\beta e_\beta \right) e_\alpha \\ &= \sum_\alpha \int dy f(y) \sum_\beta u^\beta \delta_\beta^\alpha \delta(x-y) \\ &= \sum_\alpha f(x) \cdot u^\alpha e_\alpha = f(x)u \\ &= (f \otimes u)(x). \end{aligned}$$

Furthermore, if one extends the correspondence (5.14) to elements $\psi \in \Lambda^{\alpha_0}$ by

requiring that

$$(5.16) \quad \{\psi(x)\}(\Omega_1, \dots, \Omega_\alpha) = \psi(\Omega_1, \dots, \Omega_\alpha)(x)$$

one obtains (5.13) by equating the evaluation of both sides of (5.7) for $x \in M$.

§6. COHOMOLOGY OF \mathcal{L} WITH VALUES IN LOCAL FUNCTIONALS OF CONNECTION ONE-FORMS

This section describes the cohomology which accomodates anomalies as elements of the first cohomology group. This cohomology arises from a construction analogous to the one described in section 4, but using a different representation space of the Lie algebra \mathcal{L} of the gauge group \mathcal{G} . We denote by $\Gamma^{\text{loc}}(\mathbf{A})$ the set of maps from \mathbf{A} into the reals obtained as follows: $\gamma \in \Gamma^{\text{loc}}(\mathbf{A})$ whenever

$$(6.1) \quad \gamma(a) = \int_M \bar{\gamma}(a),$$

where C is a smooth q -chain in M and $\bar{\gamma}$ is a map ⁽³¹⁾

$$(6.2) \quad a \in \mathbf{A} \rightarrow \Lambda^q(M, \mathbb{R})$$

such that, for all $a, a' \in \mathbf{A}$

$$(6.3) \quad \text{Supp}(\bar{\gamma}(a) - \bar{\gamma}(a')) \subseteq \text{Supp}(a - a'),$$

where Supp refers respectively to $\Lambda^q(M, \mathbb{R})$ in the left hand side and $hA^1(P, L)$ in the right hand side (due to Ad -equivariance, the M -support is well defined on $hA^1(P, L)$). We further define a representation ρ_Γ of \mathcal{G} on $\theta^{\text{loc}}(\alpha)$ (and an accompanying representation of \mathcal{L}), by setting

$$(6.4) \quad \{\rho_\Gamma(\Psi)\gamma\}(a) = \gamma(\rho(\Psi^{-1})a)$$

$$(6.5) \quad \rho_\Gamma(\Omega)\gamma(a) = \left. \frac{d}{dt} \right|_{t=0} \gamma(\rho(e^{-t\Omega})a).$$

We now consider the cohomology of \mathcal{L} with values in the representation space $\Gamma^{\text{loc}}(\mathbf{A})$ (another example of the general procedure already encountered in section

⁽³¹⁾ We call such maps γ *local functionals on \mathbf{A}* . Note that our definition of locality encompasses the case where a and a' , instead of being connection one-forms, are sections of a smooth fiber bundle over M (not necessarily vectorial), the support on the left hand side staying unchanged, whilst the support on the right hand side is now defined as the closure of the complement of the set $\{x \in M; a(x) = a'(x)\}$.

4). On the space

$$(6.6) \quad \left\{ \begin{array}{l} \Gamma^* = \Lambda(\mathcal{L}, \Gamma^{\text{loc}}(\mathbf{A})) = \Gamma^{\text{loc}}(\mathbf{A}) \otimes \Phi^* = \oplus_{\alpha} \Gamma^{\alpha} \\ \Gamma^{\alpha} = \Lambda^{\alpha}(\mathcal{L}, \Gamma^{\text{loc}}(\mathbf{A})) = \Gamma^{\text{loc}}(\mathbf{A}) \otimes \Phi^{\alpha} \end{array} \right.$$

direct sum of the $\Gamma^{\text{loc}}(\mathbf{A})$ -valued alternate multilinear forms on \mathcal{L} , we consider the coboundary operator δ_{Γ} given as follows: for $\gamma \in \Gamma^{\alpha}$

$$(6.7) \quad \begin{aligned} (\delta_{\Gamma} \gamma)(\Omega_0, \dots, \Omega_{\alpha}) &= \\ &= \sum_{i=0}^{\alpha} (-1)^i \rho(\Omega_i) \gamma(\Omega_0, \dots, \hat{\Omega}_i, \dots, \Omega_{\alpha}) + \\ &+ \sum_{0 \leq i < j < \alpha} (-1)^{i+j} \gamma([\Omega_i, \Omega_j], \Omega_0, \dots, \hat{\Omega}_i, \hat{\Omega}_j, \dots, \Omega_{\alpha}). \end{aligned}$$

One has $\delta_{\Gamma}^2 = 0$, $(\Gamma^*, \delta_{\Gamma})$ then defines a cohomology denoted $H^*(\mathcal{L}, \Gamma^{\text{loc}}(\mathbf{A}))$. According to Wess-Zumino compatibility anomalies are element of $H^1(\mathcal{L}, \Gamma^{\text{loc}}(\mathbf{A}))$.

§7. THE HOMOTOPY FORMULA

This section describes an algorithm (analogue to the usual Cartan Chern Weil homotopy formula) which provides a means to classify anomalies [12].

With $a \in \alpha$ and ω considered as belonging to ${}^1\mathbf{A}$ (cf. (5.2)) we set ⁽³²⁾

$$(7.1) \quad A = a + \omega$$

and consider \mathcal{F}^{tA} and \mathcal{D}^{tA} as defined in (4.71), (4.69) for the element tA of ${}^1\mathbf{A}$, $t \in [0, 1]$. We then have the limiting values

$$(7.2) \quad \left\{ \begin{array}{l} \mathcal{F}^{0A} = \mathcal{F}^0 = 0 \\ \mathcal{F}^{1A} = \mathcal{F}^A = F^a = da + \frac{1}{2} [a \wedge a] \end{array} \right.$$

and the relation

⁽³²⁾ A is thus a \mathbf{A} -connection in the sense of [4.7].

$$(7.3) \quad \mathcal{D}^{tA} A = \frac{d}{dt} \mathcal{F}^{tA}.$$

Proof. The first line of (7.2) is obvious. The second follows from the BRS relations: indeed

$$(7.4) \quad \begin{aligned} \mathcal{F}^A &= (d + s)(a + \omega) + \frac{1}{2} [a + \omega \wedge a + \omega] = \\ &= F^a + s\omega + \frac{1}{2} [\omega \wedge \omega] + sa + d\omega + [a \wedge \omega] \end{aligned}$$

where we used the fact that $[a \wedge \omega] = [\omega \wedge a]$ in the GDL $(\mathcal{A}_1^{**}, \Delta, [\wedge])$.

On the other hand we have

$$(7.5) \quad \mathcal{F}^{tA} = t\Delta a + \frac{1}{2} t^2 [A \wedge A]$$

where (6.3)

$$(7.6) \quad \frac{d}{dt} \mathcal{F}^{tA} = \Delta A + [tA \wedge A] = \mathcal{D}^{tA} A$$

From (4.53), (6.3), the Bianchi identity (4.52), and the fact that $(\mathcal{A}, \Delta, \times)$ is a GCDA, we have that

$$(7.7) \quad \begin{aligned} \mathcal{D}^{tA} (A \times \mathcal{F}^{tA} \times \mathcal{F}^{tA} \times \dots \times \mathcal{F}^{tA}) &= \\ &= \frac{d}{dt} \mathcal{F}^{tA} \times \mathcal{F}^{tA} \times \dots \times \mathcal{F}^{tA} = \\ &= \frac{1}{k} \frac{d}{dt} (\mathcal{F}^{tA} \times \mathcal{F}^{tA} \times \dots \times \mathcal{F}^{tA}). \end{aligned}$$

Applying P on the left and integrating w.r.t., t from 0 to 1, we obtain, using (4.88) and (7.2)

$$(7.8) \quad k\Delta \int_0^1 P(A \times \mathcal{F}^{tA} \times \dots \times \mathcal{F}^{tA}) dt = P(F^a \times \dots \times F^a)$$

(i.e.)

$$(7.8a) \quad \Delta Q^{2k-1} = P((F^a)^{\times k})$$

with

$$(7.9) \quad Q^{2k-1} = k \int_0^1 P(A \times \mathcal{F}^{tA} \times \dots \times \mathcal{F}^{tA}) dt.$$

Writing Q^{2k-1} as sum of homogeneous elements for ghost number and degree of form:

$$(7.10) \quad Q^{2k-1} = Q^{0,2k-1} + Q^{1,2k-2} + Q^{2,2k-3} + \dots + Q^{2k-1,0},$$

relation (6.8a) yields

$$\left\{ \begin{array}{l} \text{component } 2k, 0 : dQ^{2k-1,0} = 0 \\ \text{component } 2k-1, 1 : sQ^{2k-1,0} + dQ^{2k-2,1} = 0 \\ \text{component } 2k-2, 2 : sQ^{2k-2,1} + dQ^{2k-3,2} = 0 \\ \vdots \\ \text{component } 0, 2k : sQ^{0,2k-1} = F^a. \end{array} \right.$$

We are particularly interested in the third relation which furnishes a means of computing anomalies. Assume the principal bundle \mathbb{P} to be trivial, and let $\sigma : M \rightarrow P$ be a smooth section of \mathbb{P} . Applying the pull back by σ , which commutes with d , we obtain choosing k so that $2k - 2 = d$, the dimension of M , the following vanishing d -form M

$$(7.12) \quad \sigma^*\{(sQ^{2k-2,1})(\Omega_0, \Omega_1)\} + d\{\sigma^*Q^{2k-3,2}(\Omega_0, \Omega_1)\} = 0, \quad \Omega_0, \Omega_1 \in \mathcal{L}.$$

If we assume M compact without boundary (euclidean situation), Stokes theorem then implies

$$(7.13) \quad \int_M \sigma^*\{(sQ^{2k-2,1})(\Omega_0, \Omega_1)\} = 0, \quad \Omega_0, \Omega_1 \in \mathcal{L}.$$

Let

$$(7.14) \quad \begin{aligned} \alpha(\Omega, a) &= \int_M \sigma^*Q^{2k-2,1}(\Omega) = \\ &= \int_M \sigma^* \int_0^1 Q_t^{2k-2,1}(\Omega) dt \end{aligned}$$

with

$$(7.15) \quad Q_t = A \times (\mathcal{F}^{tA})^{\times(k-1)}, \quad A = a + \omega.$$

We now show that (7.13), which also reads (cf. 4.84)

$$(7.13a) \quad \int_M \sigma^* \int_0^1 P(sQ_t^{2k-2})(\Omega) dt = 0,$$

in fact implies the «Wess-Zumino compatibility»

$$(7.16) \quad \delta_\Gamma \alpha(\Omega_0, \Omega_1, a) = 0, \quad \Omega_0, \Omega_1 \in \mathcal{L}$$

in other words the fact that the cohomology class $[\alpha]$ of α is an element of the cohomology required for anomalies:

$$(7.16a) \quad [\alpha] \in H^1(\mathcal{L}, \Gamma^{\text{loc}}(\mathbf{A})).$$

For checking (7.16) we note that one has

$$(7.17) \quad Q_t^{2k-1,1} = \omega \times F^{ta} + (k-1)t(t-1)[a \wedge \omega] \times a \times (F^{ta})^{\times(k-2)}$$

this stemming from

$$(7.18) \quad \left\{ \begin{array}{l} A^{0,1} = \omega, \\ (\mathcal{F}^{At})^{1,1} = t(t-1)[a \wedge \omega] \\ (\mathcal{F}^{At})^{0,2} = F^{ta} = fF^a + \frac{1}{2} t(t-1)[a \wedge a]. \end{array} \right.$$

From this follows

$$(7.19) \quad \begin{aligned} Q_t^{2k-2,1}(\Omega) &= \Omega \times F^{ta} + (k-1)t(t-1)[a \times \Omega] \times a \times (F^{ta})^{\times(k-2)} \\ &= Q_t^{2k-2,1}(\Omega, a) \end{aligned}$$

hence

$$(7.20) \quad \begin{aligned} \rho(\Omega') Q_t^{2k-2,1}(\Omega) &= \\ &= Q_t^{2k-2,1}([\Omega', \Omega], a) + \frac{d}{d\lambda} \Big|_{\lambda=0} Q_t^{2k-2,1}(\Omega, \rho(e^{-\lambda\Omega'} a)). \end{aligned}$$

Therefore

$$(7.21) \quad \{sQ_t^{2k-2,1}\}(\Omega_0, \Omega_1) = Q_t^{2k-2,1}([\Omega_0, \Omega_1], a) + \\ + \frac{d}{d\lambda} \Big|_{\lambda=0} Q_t^{2k-1,1}(\Omega_1, \rho(e^{-\lambda\Omega_0}a) - \\ - Q_t^{2k-1,1}(\Omega_0, \rho(e^{-\lambda\Omega_1}a)) = -\delta_\Gamma Q_t^{2k-2,1}(\Omega, a).$$

Relation (7.16) then results from the fact that δ_Γ commutes with the operation $\int_M \sigma^* \int_0^1 dt P$, as follows from linearity.

[7.2]. *Remark.* From (6.8a) it follows that ΔQ^{2k-1} is a basic form:

$$(7.22) \quad \theta(u)\Delta Q^{2k-1} = i(u)\Delta^{2k-1} = 0, \quad u \in L.$$

This raises the question of whether this also holds for $Q^{2k-2,1}$, which would yield an intrinsic d-form q' on the base such that $Q^{2k-2,1} = \pi^*q'$ (cf. 2.7) without recourse to a section σ^* of \mathbf{P} (so without the assumption that the latter is trivial). However this does not arise ⁽³³⁾, the form $Q^{2k-2,1}$ is by definition Ad-equivariant

$$(7.23) \quad \theta(u)Q^{2k-2,1} = 0, \quad u \in L,$$

however $Q^{2k-2,1}$ is not horizontal. Indeed, we conclude from (7.17), taking account of

$$(7.24) \quad \left\{ \begin{array}{l} i(u)\omega = 0 \\ i(u)a = u1 \\ i(u)F^{ta} = i(u) \left\{ tF^a + \frac{1}{2} t(t-1)[a \cdot a] \right\} \\ \qquad \qquad \qquad = t(t-1)[u1 \cdot a] \end{array} \right. =$$

that one has

$$(7.25) \quad \frac{i(u)Q_t^{2k-2,1}}{t(t-1)} = \\ = (-\omega \times [u1 \cdot a] + (k-1)[u1 \times \omega] \times a + (k-1)[a \cdot \omega] \times a) \times (F^{ta})^{\times(k-1)} + \\ + (k-2)(\omega \times F^{ta} + (k-1)t(t-1)[a \cdot \omega] \times [u1 \cdot a]) \times (F^{ta})^{\times(k-1)}$$

In particular, for $k = 2$

⁽³³⁾ No more that in the case of the Chern-Simons form TP . What we have here is an analogy to the Chern-Simons situation, with the replacement $a \rightarrow A$ (cf. [9]).

$$(7.26) \quad \begin{aligned} i(u)Q_t^{2k-2,1} &= -\omega \times [u1 \times a] + [u1 \wedge \omega] \times a + [a \wedge \omega] \times a = \\ &= [u1 \wedge (\omega \times a)] - 2\omega \times [u1 \wedge a] + [a \wedge \omega] \times \omega. \end{aligned}$$

Upon application of P the first term r.h.s. vanishes (cf. (4.90), but the two following terms persist, yielding a non vanishing result.

It is easy to find a substitute for Q^{2k-1} whose terms $Q^{s,2k-1-s}$ are basic, horizontal, replacing the family tA by the family $tA + (1-t)a$, where a is a fixed background connection on \mathbf{P} , which then need not be assumed to be trivial [8]. The algebraic constructions of this paper have to be generalized as will be done in a forthcoming article.

APPENDIX A. GRADED DIFFERENTIAL ALGEBRAS

[A.1]. DEFINITIONS. (i) A GDA (*graded differential algebra*) is a graded real vector space $\mathbf{A} = \bigoplus_{n \in \mathbf{N}} \mathbf{A}^n$, equipped with a bilinear product $\mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ and a linear operator $d : \mathbf{A} \rightarrow \mathbf{A}$ (the derivative), with the properties

$$(A.1) \quad \mathbf{A}^p \cdot \mathbf{A}^q \subset \mathbf{A}^{p+q}, \quad p, q \in \mathbf{N}$$

$$(A.2) \quad d\mathbf{A}^p \subset \mathbf{A}^{p+1}, \quad p \in \mathbf{N}$$

$$(A.3) \quad d(ab) = Da \cdot b + (-1)^p da \cdot b, \quad a \in \mathbf{A}, b \in \mathbf{A}^p, p \in \mathbf{N}$$

$$(A.4) \quad d^2 = 0.$$

(ii) A GCDA (*graded-commutative differential algebra*)⁽³⁴⁾ is a GDA \mathbf{A} with an associated and «graded-commutative» product:

$$(A.5) \quad a \cdot (bc) = (a \cdot b)c, \quad a, b, c \in \mathbf{A}$$

$$(A.6) \quad b \cdot a = (-1)^{p \cdot q} a \cdot b, \quad a \in \mathbf{A}^p, b \in \mathbf{A}^q, p, q \in \mathbf{N}.$$

(iii) A DGL⁽³⁴⁾ (*differential graded Lie algebra*) is a GDA with product (2) a «graded Lie bracket»:

$$(A.7) \quad b \cdot a = -(-1)^{pq} a \cdot b, \quad a \in \mathbf{A}^p, b \in \mathbf{A}^q$$

$$(A.8) \quad \begin{aligned} (-1)^{pr} a \cdot (b \cdot c) + (-1)^{qp} b \cdot (c \cdot a) + (-1)^{rq} c \cdot (a \cdot b) &= 0; \\ a \in \mathbf{A}^p, b \in \mathbf{A}^q, c \in \mathbf{A}^r. \end{aligned}$$

⁽³⁴⁾ In concrete examples as those encountered in the text, it is natural to denote the product of a GCDA by a wedge-like symbol, and that of a DGL by a bracket-like symbol.

(iv) Let L be a Lie algebra and α be a GDA. An *action of L on α* is a pair (θ, i) of linear maps from L to the linear operators of α with the properties (35)

$$(A.9) \quad \left\{ \begin{array}{l} \theta(u)\mathfrak{A}^p \subset \mathfrak{A}^p, \quad p \in \mathbb{N} \\ \theta(u)(a \cdot b) = \theta(u)a \cdot b + a \cdot \theta(u)b, \quad a, b \in \mathfrak{A}, \quad u \in L \end{array} \right.$$

$$(A.10) \quad \left\{ \begin{array}{l} i(u)\mathfrak{A}^p \subset \mathfrak{A}^{p-1}, \quad p \in \mathbb{N}, \quad i(u)|_{\mathfrak{A}^0} = 0, \quad u \in L \\ i(u)(a \cdot b) = i(u)a \cdot b + (-1)^p a \cdot i(u)b, \quad a \in \mathfrak{A}^p, \quad b \in \mathfrak{A} \end{array} \right.$$

and

$$(A.11) \quad i(u)^2 = 0, \quad u \in L$$

$$(A.12) \quad \theta([u, v]) = \theta(u)\theta(v) - \theta(v)\theta(u), \quad u, v \in L$$

$$(A.13) \quad \theta(u)i(v) - i(v)\theta(u) = i([u, v]), \quad u, v \in L$$

$$(A.14) \quad \theta(u) = i(u)d + di(u), \quad u \in L$$

Note that these properties imply

$$(A.15) \quad \theta(u)d = d\theta(u) (= di(u)d), \quad u \in L$$

$$(A.16) \quad \theta(u)i(u)\theta(u), \quad u \in L.$$

(In fact (A.12) follows from (A.14) and the fact that $d^2 = 0$).

[A.2]. *Skew tensor products of GDAs.*

Let $(\mathfrak{A} = \bigoplus_{p \in \mathbb{N}} \mathfrak{A}^p, d, \cdot)$ and $(\Psi = \bigoplus_{\alpha \in \mathbb{N}} \Psi^\alpha, \delta, \circ)$ be two real (36) GDAs. Their *skew product* as graded algebras is the usual tensor product of vector spaces

$$(A.17) \quad \Pi = \mathfrak{A} \otimes \Psi$$

equipped with the bilinear skew product determined by

(35) (A.9) states the fact that $\theta(u), u \in L$, is a 0-grade derivation of α ; and (A.10) that $i(u), u \in L$, is a -1 -graded derivation of α .

(36) Or for that matter complex.

$$(A.18) \quad (a \otimes \varphi) \Delta((b \otimes \psi) = (-1)^{\alpha a} (a \cdot b) \otimes (\varphi \circ \psi), \quad \begin{cases} a \in \mathbf{A}, & b \in \mathbf{A}^\rho \\ \varphi \in \Psi^\alpha, & \psi \in \Psi \end{cases}$$

and the grading

$$(A.19) \quad \Pi^n = \sum_{p+\alpha=n} \mathbf{A}^p \otimes \Psi^\alpha.$$

We furthermore consider the following operators on Π :

$$(A.20) \quad D = d + \sigma$$

where ⁽³⁷⁾

$$(A.21) \quad d = d \otimes 1_\Psi$$

$$(A.22) \quad \sigma = (-1)^\partial \otimes \delta$$

where $(-1)^\partial$ denotes the grading in relation of \mathbf{A}

$$(A.23) \quad (-1)^\partial = \begin{cases} \text{id} & \text{on } \mathbf{A}^{2p} \\ -\text{id} & \text{on } \mathbf{A}^{2p+1}, \quad p \in \mathbf{N} \end{cases}$$

and \otimes r.h.s. of (A.21), (A.22) denotes a standard tensor product of linear operators ⁽³⁹⁾

$$(A.24) \quad (A \otimes B)(a \otimes \psi) = (Aa) \otimes (B\psi), \quad a \in \mathbf{A}, \quad \psi \in \Psi.$$

We recall that these definitions imply the following facts:

(i) With the product (A.18) and the grading (A.19) Π is a graded algebra, i.e. $\Pi^n \cdot \Pi^m \subset \Pi^{n+m}$, $n, m \in \mathbf{N}$.

(ii) d and σ are graded derivations of Π of grade 1 ⁽³⁸⁾. Moreover d and σ anticommute:

⁽³⁷⁾ Using the same symbol for d acting on α and $d = d \otimes 1_\Psi$ acting on Π should cause no confusion.

⁽³⁸⁾ A linear operator L on Π is of grade r , $r \in \mathbf{N}$ whenever $L\Pi^n \subset L\Pi^{n+r}$, $n \in \mathbf{N}$ ($\Pi^k = 0$, $k < 0$). It is a *graded derivation* wherever $L(\alpha \cdot \beta) = (L\alpha) \cdot \beta + (-1)^r \alpha \cdot L\beta$, $\alpha \in \Pi^p$, $\beta \in \Pi$.

⁽³⁹⁾ In fact, (A.21) and (A.22) could be written $d = d \otimes 1_\Psi$, $\sigma = 1_\alpha \otimes \delta$, with \otimes a *graded tensor product* of operators.

$$(A.25) \quad d\sigma + \sigma d = 0.$$

Consequently, D is a graded derivation of Π of grade -1 and vanishing square

$$(A.26) \quad D^2 = 0,$$

making (Π, δ, \cdot) a GDA.

(iii) If (\mathbf{A}, d, \cdot) and (Ψ, δ, \circ) are GCDA, (Π, D, Δ) is a GCDA.

(iv) If (\mathbf{A}, d, \cdot) is a DGL and (Ψ, δ, \circ) is a GCDA, (Π, D, Δ) is a DGL.

Proof. (i): obvious.

(i) We have, for $a \in \mathbf{A}^p, b \in \mathbf{A}^q, \varphi \in \Psi^\alpha, \psi \in \Psi^\beta$

$$(A.27) \quad \begin{aligned} d\{(a \otimes \varphi)\Delta(b \otimes \psi)\} &= (-1)^{\alpha q} d\{(a \cdot b) \otimes (\varphi \circ \psi)\} = \\ &= (-1)^{\alpha q} (da \cdot b + (-1)^p a \cdot db) \otimes (\varphi \circ \psi) = \\ &= (-1)^{\alpha q + \alpha q} (da \otimes \varphi)\Delta(b \otimes \psi) + (-1)^{\alpha q + p + (q+1)} (a \otimes \varphi)\Delta(db \otimes \psi) = \\ &= \{d(a \otimes \varphi)\}\Delta(b \otimes \psi) + (-1)^{p+\alpha} (a \otimes \varphi)\Delta d(b \otimes \psi) \end{aligned}$$

$$(A.28) \quad \begin{aligned} \sigma\{(a \otimes \varphi)\Delta(b \otimes \psi)\} &= (-1)^{\alpha q} \sigma\{(a \cdot b) \otimes (\varphi \circ \psi)\} = \\ &= (-1)^{\alpha q + p + q} (a \cdot b) \otimes (\sigma\varphi \circ \psi + (-1)^\alpha \varphi \circ \sigma\psi) = \\ &= (-1)^{\alpha q + p + q + (\alpha+1)q} (a \otimes \sigma\varphi)\Delta(b \otimes \psi) + (-1)^{\alpha q + p + q + \alpha + \alpha q} (a \otimes \varphi)\Delta(b \otimes \sigma\psi) \\ &= \{\sigma(a \otimes \varphi)\}\Delta(b \otimes \psi) + (-1)^{p+\alpha} (a \otimes \varphi)\Delta\sigma(b \otimes \psi) \end{aligned}$$

$$(A.29) \quad \begin{aligned} d\sigma(a \otimes \varphi) &= (-1)^p d(a \otimes \delta\varphi) = (-1)^p (da \otimes \delta\varphi) = \\ &= (-1)^{p+p+1} \sigma(da \otimes \varphi) = -\sigma d(a \otimes \varphi) \end{aligned}$$

(iii) We have, for a, b, φ, ψ as above and $c \in \mathbf{A}^r, \theta \in \Psi^\gamma$

$$(A.30) \quad \begin{aligned} (a \otimes \varphi)\Delta\{(b \otimes \psi)\Delta(c \otimes \theta)\} &= (-1)^{\beta r} (a \otimes \varphi)\Delta\{(b \cdot c) \otimes (\psi \circ \theta)\} = \\ &= (-1)^{\beta r + \alpha(q+r)} (a \cdot b \cdot c) \otimes (\varphi \circ \psi \circ \theta) = \\ &= (-1)^{\beta r + \alpha(q+r) + r(\alpha+\beta)} \{(a \cdot b) \otimes (\varphi \circ \psi)\}\Delta(c \otimes \theta) = \\ &= (-1)^{\alpha q + \alpha q} \{(a \otimes \varphi)\Delta(b \otimes \psi)\}\Delta(c \otimes \theta) \end{aligned}$$

$$(A.31) \quad \begin{aligned} (b \otimes \psi)\Delta(a \otimes \varphi) &= (-1)^{\beta p} (b \cdot a) \otimes (\psi \circ \varphi) = \\ &= (-1)^{\beta p + p q + \alpha \beta} (a \cdot b) \otimes (\varphi \circ \psi) = \\ &= (-1)^{(\alpha+\beta)(p+q)} (a \otimes \varphi)\Delta(b \otimes \psi) \end{aligned}$$

(iv) Let $a, b, c, \varphi, \psi, \theta$, be as above. For the commutation of $a \otimes \varphi$ and $b \otimes \psi$, we have the same computation as in (A.31), with the alteration that, now, $b \cdot a =$

$= -(-1)^{pq} a \cdot b$, whence an overall sign, leading to (A.7). We check (A.8): we have

$$\begin{aligned}
 \text{(A.32)} \quad & (-1)^{(p+\alpha)(r+\gamma)}(a \otimes \varphi) \Delta \{(b \otimes \psi) \Delta (c \otimes \theta)\} = \\
 & = (-1)^{(p+\alpha)(r+\gamma) + \beta r} (a \otimes \varphi)(b \cdot c) \otimes (\psi \circ \theta) = \\
 & = (-1)^{(p+\alpha)(r+\gamma) + \beta r + \alpha(q+r)} (a \cdot b \cdot c) \otimes (\varphi \circ \psi \circ \theta) = \\
 & = (-1)^{\alpha q + \beta r + \gamma p} (-1)^{pr + \alpha \gamma} (a \cdot b \cdot c) \otimes (\varphi \circ \psi \circ \theta).
 \end{aligned}$$

Since $\alpha q + \beta r + \gamma p$ is invariant under circular permutation, the graded Jacobi identity for Π is a consequence of that for α , given that $(-1)^{\alpha \gamma} \varphi \circ \psi \circ \theta$ is invariant under circular permutations, a straightforward consequence of the GCDA nature of Ψ .

Remark. (i) The proof of (i) applies to the more general situation where d and δ are graded derivation of α , resp. Ψ , of odd grade p , resp. q . Defining d and σ on the skew product Π as in (A.21), (A.22) the latter are still mutually anticommuting derivations of Π , of respective grades p and q .

Indeed, in the proofs (A.27), (A.28), (A.29), no use is made of the fact that $d^2 = 0$, $\delta^2 = 0$, and the grade of d and δ enters only through its parity.

APPENDIX B. «COVARIANT DERIVATIVES» IN DGLs

PROPOSITION. *Let $(L = \bigoplus_{n \in \mathbb{N}} L^{(n)}, d, [\cdot])$ be a DGL. For $\alpha \in L^{(1)}$, $F^\alpha \in L^{(2)}$ and the map $D^\alpha : L \rightarrow L$ are defined as follows:*

$$\text{(B.1)} \quad F^\alpha = d\alpha + \frac{1}{2} [\alpha \wedge \alpha]$$

$$\text{(B.2)} \quad D^\alpha \lambda = d\lambda + [\alpha \wedge \lambda].$$

We then have that

(i) D^α is a graded derivation of L

$$\text{(B.3)} \quad D^\alpha [\lambda \wedge p] = [(D^\alpha \lambda) \wedge p] + (-1)^n [\lambda \wedge D^\alpha p], \quad \lambda \in L^{(n)}, \quad p \in L,$$

whose square is given by

$$\text{(B.4)} \quad (D^\alpha)^2 \lambda = [F^\alpha \wedge \lambda], \quad \lambda \in L$$

(ii) *We have the Bianchi identity*

$$\text{(B.5)} \quad D^\alpha F^\alpha = 0.$$

(iii) *The map $[\alpha \wedge \cdot]$:*

$$(B.7) \quad \lambda \in L \rightarrow [\alpha \wedge \lambda] \in L$$

is a graded derivation of L ; in fact D^α is the sum of d and $[\alpha \wedge \cdot]$ (generally for $\alpha \in L^{(p)}$ $[\alpha \wedge \cdot]$ is a derivation of order p in the sense that

$$(B.8) \quad [\alpha \wedge [\lambda \wedge \mu]] = [[\alpha \wedge \lambda] \wedge \mu] + (-1)^{np}[\lambda \wedge [\alpha \wedge \mu]], \quad \lambda \in L^p, \mu \in L^{(q)}.$$

Proof. (i) d is by definition a graded derivation of $([L, \wedge])$; thus the first assertion in (i) is a consequence of (iii), which in turn follows from the following special case of the graded Jacobi identity: for $\lambda \in L^{(p)}$, $\mu \in L^{(q)}$, $\alpha \in L^{(n)}$

$$(B.9) \quad (-1)^{nq}[\alpha \wedge [\lambda \wedge \mu]] + (-1)^{np}[\lambda \wedge [\mu \wedge \alpha]] + (-1)^{pq}[\mu \wedge [\alpha \wedge \lambda]]$$

with the commutation properties

$$(B.10) \quad [\mu \wedge \alpha] = -(-1)^{nq}[\alpha \wedge \mu]$$

$$(B.11) \quad [\mu \wedge [\alpha \wedge \lambda]] = -(-1)^{q(n+p)}[\alpha \wedge \lambda] \wedge \mu$$

implying

$$(B.12) \quad [\alpha \wedge [\lambda \wedge \mu]] = [[\alpha \wedge \lambda] \wedge \mu] + (-1)^{np}[\lambda \wedge [\alpha \wedge \mu]], \quad \lambda \in L^{(p)}.$$

We now check (A.4): we have, taking now $n = 1$

$$(B.13) \quad (D^\alpha)^2 \lambda = d(d\lambda + [\alpha \wedge \lambda]) + \alpha(d\lambda + [\alpha \wedge \lambda]) = \\ = [d\alpha \wedge \lambda] + [\alpha \wedge [\alpha \wedge \lambda]]$$

however, using (A.10)

$$(B.14) \quad [\alpha \wedge [\alpha \wedge \lambda]] = \frac{1}{2} [\alpha \wedge [\alpha \wedge \lambda]] + [[\alpha \wedge \alpha] \wedge \lambda] - [\alpha \wedge [\alpha \wedge \lambda]] = \\ = \frac{1}{2} [[\alpha \wedge \alpha] \wedge \lambda].$$

(ii) We have

$$(B.15) \quad D^\alpha f^\alpha = dF^\alpha + \alpha \wedge F^\alpha = d d\alpha + \frac{1}{2} [\alpha \wedge \alpha] + \alpha \wedge d\alpha + \frac{1}{2} [\alpha \wedge \alpha] \\ = \frac{1}{2} [d\alpha \wedge \alpha] - \frac{1}{2} [\alpha \wedge d\alpha] - [\alpha \wedge d\alpha] = 0.$$

where $[d\alpha \wedge \alpha] = -[\alpha \wedge d\alpha]$, due to the commutation rule, and $[\alpha \wedge [\alpha \wedge \alpha]] = 0$, to the graded Jacobi identity.

(iv) Immediate from (B.4) and (B.5). ■

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